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Democracy between anarchy and dictatorship: A fragile balance

Working Paper 1

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Abstract

Starting from observations and basic questions as posed by Acemoglu and Robinson in their work on the "narrow corridor", we investigate the situation of liberal democracies using a dynamic model originating from biology, the Holling type III functional response, augmented by negative external effects, between two groups in a society, the elite and civil society. We analyze the dynamics of an uncontrolled system, a system with one-sided control by the elite, and a game model with strategic interactions between the two groups using dynamic systems theory, bifurcation theory, optimal control theory, and dynamic game theory. A low prevalence of negative effects between groups, a low inequality of initial endowments, and a low discount rate of future events are helpful for establishing a liberal democracy. The competitive dynamic game model seems to grant slightly better chances for a stable democracy.

Keywords: Political economy; democracy; dictatorship; anarchy; dynamic system; bifurcations; optimal control and dynamic games.

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1 Introduction

Over the last 20 years or so, liberal democracy[1] worldwide has faced increasing challenges from economic shocks (the Great Recession, i.e., the financial and economic crisis of 2007-2010), the COVID-19 pandemic, and wars (the Russian invasion of Ukraine, the Hamas-Israel war), among others, which were related to the emergence of populist and extremist political movements as well as doubts about the ability of democratic governments to deal with these problems. In general, democracy seems to be in retreat[2], and autocratic regimes are on the rise, blighting hopes for a global democratic future after the demise of Communism, as expressed, for instance, by Fukuyama [21]. Moreover, several attempts to topple corrupt and autocratic regimes, such as the Arab Spring, resulted in anarchy and civil wars or in new dictatorships (see, e.g., Boucekkine et al. [12], Feichtinger et al. [20]). For those concerned with the preservation of freedom and human rights, the question arises as to whether we have to expect the end of liberal democracy rather than "The End of History".

Based on their earlier seminal work about the failure or flourishing of states worldwide (Acemoglu and Robinson [2, 3]), Acemoglu and Robinson [4, 5, 6] suggested a model of what they call the "narrow corridor" between autocracy and anarchy in which liberal democracies move and are at risk of developing into one of those extreme forms of government. Using historical evidence, they also identified which features of a political system contribute to the robustness of liberal democracy. Their model rests, of course, on a number of special assumptions, which raises the question as to whether their results also hold under alternative ones. Here we investigate this question by using different dynamics, based on the Holling type III function originating in biology and including the possibility of adverse interactions between groups in a society, and examine the behavior of a dynamic system otherwise similar to that of Acemoglu and Robinson [4, 6]. We show that in this case, a greater variety of equilibria and paths (both in the transitional period as well as in the long run) can occur and provide interpretations of the results in terms of the model parameters.

Our approach is in the tradition of analyzing cultural evolution using population dynamics models from mathematical biology; see Boyd and Richerson [13], Richerson and Boyd [34], Efferson et al. [19]. The work of Turchin belongs to this strand of literature, who, as a biologist, uses the scientific apparatus of mathematical biology to develop cliodynamics to explain long-run political and economic developments. This is a field at the intersection of cliometrics (economic history with econometric models), historical macrosociology and

[1]A liberal democracy is characterized by the rule of law, checks and balances, limitations on government, the protection of individual rights and free and fair elections. It allows for productive cooperation between the different groups (in our case, elite and civil society), which have similar capacities due to a relatively equal distribution of initial endowments. It corresponds to Popper's concept of an Open Society (Popper [32]).

[2]The annual democracy index, which provides a snapshot of the state of global democracy, registered a decline in its global score from 5.29 in 2022 to 5.23 in 2024. See https://www.eiu.com/n/democracy-index-conflict-and-polarisation-drive-a-new-low-for-global-democracy. Also, the Democracy Report 2025 of the V-DEM Institute [41] states that the world has fewer democracies ($N = 88$) than autocracies ($N = 91$) for the first time in over 20 years. In particular, liberal democracies have become the least common regime type in the world. Nearly 3 out of 4 persons in the world now live in autocracies, which is the highest ratio since 1978. More evidence for the tendency towards increased authoritarianism, explanations thereof and how to counter it can be found in Applebaum [8] or Levitsky and Ziblatt [30], for instance.

mathematical modeling of social processes. Turchin [37] developed a model of territorial dynamics of agrarian states and explained various historical cycles of expansion – stagflation – crisis – depression (Turchin and Nefedov [40]) as well as the rise and fall of empires (Turchin [38]). Although many historians doubt whether history follows mathematical dynamics,[3] Turchin [39] forecasted successfully the 2020s to be a decade of increasing force, especially in the United States.

This paper is structured as follows: Section 2 describes our model. Section 3 then presents results for an uncontrolled dynamic system, that is, one where both groups (agents) act in accordance with the dynamic system, which is governed without their attempts to actively influence the dynamics. Several patterns of development are possible, depending on the parameters. In Section 4, we use optimal control theory and consider the case of an optimizing government (the elite) with production in the objective function and a passive (adapting) civil society. We outline different regimes depending on the key parameters, using bifurcation analysis. Section 5 formulates a dynamic game with the elite and civil society as players and shows how the strength of the adverse effects of each player on the other influences the results. In the uncontrolled system, the controlled system, and in the game, a "conflict parameter" expressing the power to reduce the other group's capacity (strength) is crucial for the existence, the shape, and the "narrowness" of the "corridor", the area in which a liberal democracy can flourish. Section 6 concludes.

2 The model

Following Acemoglu and Robinson [4, 6], we assume that the political system under consideration consists of two groups acting as homogeneous agents: the elite and civil society. The elite can also be interpreted as the public sector in the economic system or, as an agent, the government while civil society corresponds to the private sector. At each time t, assumed to be continuous, the elite has capacity $x(t)$ as its disposal, and civil society capacity $y(t)$. These capacity variables can be interpreted as the "power" of the agent concerned, which may depend, among others, on the number of individuals within the group, their productive possibilities, and their endowments. The elite can change its capacity by efforts denoted by the variable $u(t)$, which entails some cost. For civil society, we first assume that it reacts passively to the actions of the governing elite. Alternatively, we assume that it can also increase its capacity by an effort variable $v(t)$. Changes in efforts can also be interpreted as investments in the agents' capacities or human capital. Each capacity is subject to obsolescence or depreciation of human capital with rates d_x and d_y respectively.

In the basic model, we consider the following dynamic system, describing the development of the capacity variables:

$$\dot{x}(t) = \phi(x(t))\,\psi(y(t))\,u(t) - d_x x(t), \tag{1}$$

$$\dot{y}(t) = \phi(y(t))\,\psi(x(t))\,v(t) - d_y y(t). \tag{2}$$

[3] See the reviews by Tainter [35] and Zhao [42]. Already Popper [33] argued strongly against predictability of history. Thus, even political philosophy is involved in this interdisciplinary approach.

In the following, for the sake of brevity we omit the time argument. For the function $\phi(x)$, we consider a convex-concave shape with $\phi(0) = 0$, $\phi'(x) > 0$ while $\psi(x)$, in contrast, decreases in a convex-concave shape. One such form here is the Holling type III function (as discussed below), i.e., $\phi(x) = a_x \left(x^2/(c_x + x^2)\right)$, and $\psi(x) = 1 - b_x \left(x^2/(c_x + x^2)\right)$, and analogously for y. This is plotted in Figure 1. Naturally we have $\lim_{x\to\infty} \phi(x) = a_x$ and $\lim_{y\to\infty} \phi(y) = a_y$ which is $a_x = a_y = a = 1$ in the figure; equally $\lim_{x\to\infty} \psi(x) = 1 - b_x$ and $\lim_{y\to\infty} \psi(y) = 1 - b_y$.

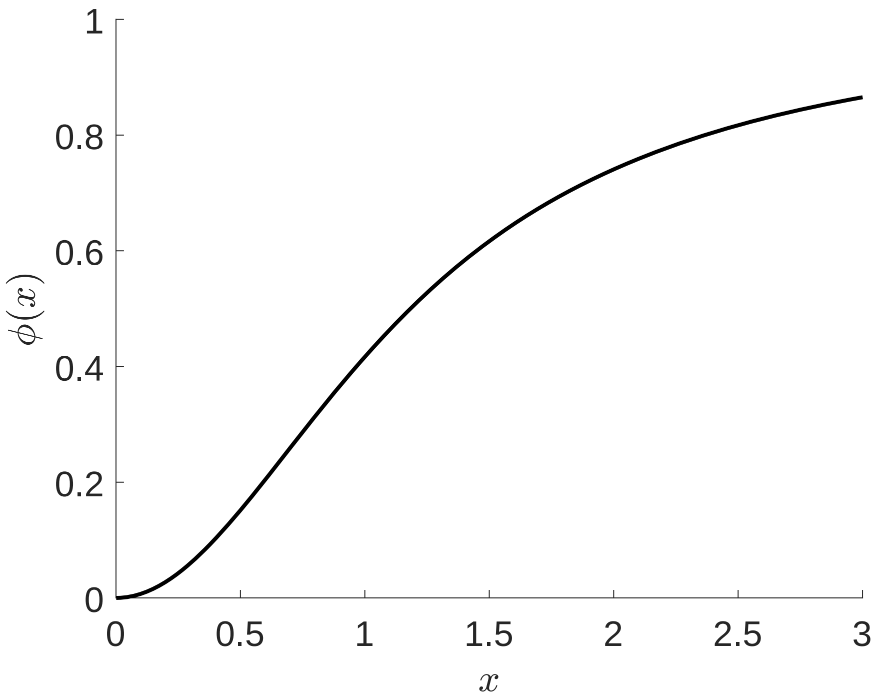


Figure 1: Function $\phi(x)$ as Holling type III with $a = 1$ and $c_x = 1.4$.

The Holling type III function was introduced by Holling [26, 27] for predator-prey systems in ecology to describe a nonlinear relation between the density of the prey population and the number of prey items consumed. In the context of our model, this sigmoid can be interpreted as a learning curve: At low levels of the capacity of the group, for instance in a situation where the group has just been formed and does not have much experience with the political system (which, for example, has been recently established after a regime change, or if both groups are not yet strongly established), the group has low capacity. This changes with an increase in its capacity, first slowly and then, after crossing some threshold, more rapidly up to saturation level, when the group cannot learn any more as it is fully aware of the workings of the political system and the latter is strongly established. The convex part captures the increasing returns that Acemoglu and Robinson emphasize but smoothly instead of non-differentiability that is crucial for their model.

As with Acemoglu and Robinson, we do not discuss the process of group formation and assume that the two groups act as coherent agents; hence the analysis may be criticized from the point of view of methodological individualism for lacking some micro foundation. If we look at the use of Holling's type III functional responses in biology, work by Dawes and Souza [14] provides a derivation of these functions (including type III) from the (genetically determined) behavior of individual predators and the demographic characteristics of predators and prey. The Holling type III form was found to be characteristic for generalist predators (Turchin [36], p. 83). More generally, Kalinkat et al. [28] found that learning, adaptive foraging, and prey switching can lead to the establishment of the Holling type III mechanism.

The analogy between a predator-prey system and the interaction of the two groups in our model society can be underpinned by noting that the elite and civil society are, to some extent, simultaneously both predator and prey, competing against each other for the scarce resources in society. In addition to this antagonism between the two groups, we have a potential for cooperation between them, with the joint interest of increasing available resources when it comes to producing output in addition to home production, which requires private and public capital by civil society and the elite (the government) respectively. This feature is captured by making changes in the capacities of both groups depending positively on their own capacity level and negatively on that of the other group. In economic terms, there are positive and negative externalities between the two groups. In contrast to a predator-prey system, each group can continue to exist when the other group is extinct (has zero capacity), but then it cannot produce any output beyond its subsistence level (by home production, for instance).

In particular, we assume that each group's investment in its capacity depends positively on its own capacity and negatively on the other group's capacity through the function $\psi(\cdot)$, which implies a strong element of conflict between the two groups. Parameter b ($b_x = b_y = b$ in the symmetric case) shows the negative influence of the strength of each group on the other group's capacity and is positive. We call it the conflict parameter; it turns out that it is crucial for the dynamic results. In addition, we assume that the society produces output above some subsistence level (poverty line) according to a Cobb-Douglas production function $f(x, y) = Ax^{\alpha}y^{1-\alpha}$, $0 < \alpha < 1$, to which both groups contribute with public (x) and private (y) capital (including human capital) as production factors, assumed to be equal to our capacity variables. The development of the capacity of each of the groups does not depend directly on the output, which is an element of joint interest for both groups in the model.

3 The uncontrolled solution

3.1 Phase portraits

First, we consider the uncontrolled solution of the system, that is, $u \equiv v \equiv 1$ for all t. This means that the two groups do not actively react to the other group's actions and do not try to steer the system on their own but adapt to the development of the system passively. This may be the case when the political system is such that the two groups have roles pre-assigned through customs and tradition or a long-run stable institutional environment, as was the case in pre-historic tribal societies or in some societies in ancient times, for instance. Alternatively, the feedback laws in (3) describe the agents' actions as heuristic or optimal for unspecified preferences. In this case, system (1)-(2) becomes:

$$\begin{aligned}\dot{x} &= a_x \frac{x^2}{c_x + x^2}\left(1 - b_y \frac{y^2}{c_y + y^2}\right) - d_x x,\\ \dot{y} &= a_y \frac{y^2}{c_y + y^2}\left(1 - b_x \frac{x^2}{c_x + x^2}\right) - d_y y.\end{aligned} \tag{3}$$

Figure 2 is a simple illustration of this descriptive dynamics presented as a phase diagram. Here and in the rest of Section 3, we omit the constant control variables for ease of notation. We assume the following numerical values of the parameters: $a_x = a_y = a = 0.5$, $b_x = 0.02$, $b_y = 0.01$, $c_x = c_y = 1.4$, $d_x = 0.17$, $d_y = 0.2$.

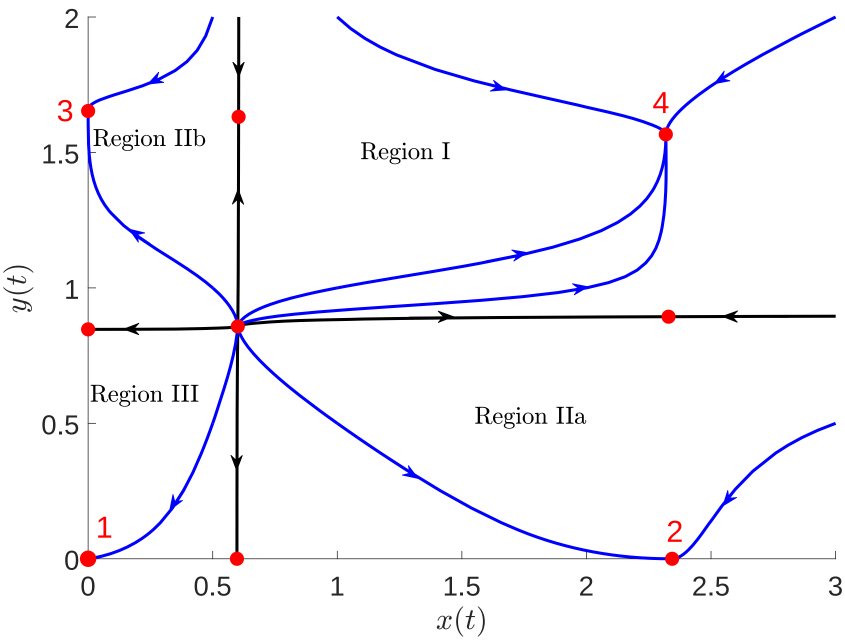


Figure 2: Phase diagram of system 3. Parameters: $a_x = a_y = a = 0.5$, $b_x = 0.02$, $b_y = 0.01$, $c_x = c_y = 1.4$, $d_x = 0.17$, $d_y = 0.2$.

Here we find nine equilibria (red points): The origin is a stable node, the low inner equilibrium is an unstable node, the high inner equilibrium is a stable node, the two low boundary equilibria are saddle points, the two high boundary equilibria are stable nodes, and the two asymmetric inner equilibria are saddle points. Neglecting the unstable equilibria, we have four long-run outcomes (marked by corresponding red numbers in Figure 2): 1. the origin, 2. a boundary equilibrium on the x-axis, 3. a boundary equilibrium on the y-axis, and 4. the high inner equilibrium. They imply the following regions of attraction (areas of combinations of both groups' capacities (x, y)), as shown in Figure 2:

- Region I: The dynamic system moves toward equilibrium 4. This implies an inclusive state in the sense of Acemoglu and Robinson or – in the case of a democratic system – a liberal democracy. Here both groups have positive capacities and contribute positively to the output, and their joint effort leads to a well-designed liberal democratic state.
- Region IIa: The dynamic system moves toward equilibrium 2. Here the elite has high capacity, and civil

society has none. This is the case in a totalitarian dictatorship like Hitler's, Stalin's, and probably Mao's terror regimes, or North Korea today. Such a situation is highly unwelcome: Nobody (except for the elite, the few members of the dictator's inner circle) has a good life, nothing of value is produced beyond the subsistence level, and the government's main function (in the hands of the elite) is to maintain its power. The entire society, except for the dictator's sycophants, is characterized by forced labor, and all public resources are diverted to suppress any resistance or to wage war against the regime's own population or that of other countries.

- Region IIb: The dynamic system moves toward equilibrium 3. The private sector (civil society) has high capacity while the elite or the government has none and is ineffective. This is the unproductive situation of anarchy, such as in Somalia since the 1990s, for instance. Society is driven by egoistical behavior, lacking an effective legal system and public infrastructure; hence nothing is produced (except for what its members need, which may be acquired by piracy, robbery, or theft, etc.). Some observers may identify this situation with an extreme variant of a libertarian society with a "neoliberal" economy. Although no serious free-market advocate would find a situation with no public sector at all attractive, the possibility of a society with very low initial capacity of the elite (the government) resulting in such an equilibrium cannot be completely discarded as a possibility.

- Region III: The dynamic system moves toward equilibrium 1. This may be identified with Thomas Hobbes's (2012) idea of everybody being everybody's wolf, resulting in a "solitary, poor, nasty, brutish, and short" life for everybody, his "war of all against all". Needless to say, nothing of value is produced here either. The outcome is similar to but even worse than the situation in regions IIa and IIb. Hobbes has only seen region III as his "state of nature" (his "Behemoth") and regions I and IIa as the "Leviathan", based on a "social contract" avoiding the anarchy of the "Behemoth". The present model emphasizes, however, that without limits being placed on the "Leviathan" (the elite or the government), the outcome is also unwelcome (region IIa). The "Behemoth" can be interpreted as poverty line situation, where both groups have minimal capacities (at the subsistence level).

It turns out that the only desirable equilibrium from the point of view of democracy is the high inner one. Movements towards this equilibrium (region I) correspond to Acemoglu and Robinson's "narrow corridor". However, as Figure 2 clearly shows, the "corridor" need not be narrow and it need not be a "corridor" at all. If both groups in a society have sufficient capacities that do not differ too much from one another, the result may be a robust long-run outcome that would not easily be shattered by exogenous shocks. Of course, this result depends on the parameters of the model, hence the sensitivity of that case must be examined with respect to the parameters.

To do so, we start with the symmetric case, which we take as a benchmark case for the comparisons with a few alternative specifications. Assume the following parameter values of the model: $a_x = a_y = a = 0.5$, $b_x = b_y = b = 0.01$, $c_x = c_y = c = 1.4$, $d_x = d_y = d = 0.2$. Figure 3 presents the phase diagram and the Holling type III function for this system. It is qualitatively similar to the previous situation, albeit with a smaller area for the inclusive state. Due to its symmetry, the "corridor" is again symmetric but still open for sufficiently high initial values of the capabilities for both groups.

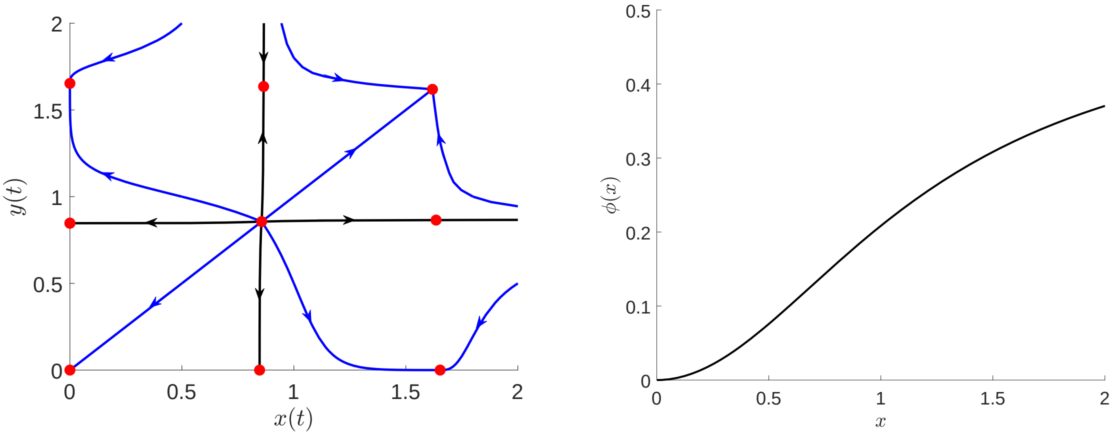


Figure 3: Phase diagram of system 3. Parameters: $a = 0.5$, $b = 0.01$, $c = 1.4$, $d = 0.2$.

The area of the inclusive state can be enlarged when the learning curves become steeper (a larger parameter c). This can be seen in Figure 4, where $a = 0.5$, $b = 0.01$, $c = 0.5$, and $d = 0.2$.

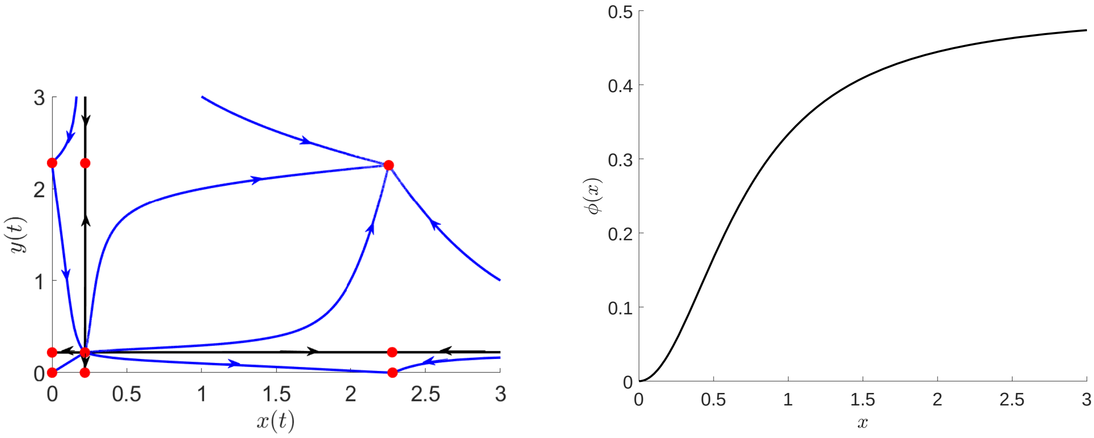


Figure 4: Phase diagram of system 3. Parameters: $a = 0.5$, $b = 0.01$, $c = 0.5$, $d = 0.2$.

One possible intuition of this result is as follows: When both groups learn faster, the area of the inclusive state becomes larger. One may conjecture that political education for both the elite and civil society increases the chances of a liberal democracy. Of course, this is true only if the content of that education is not dominated by one-sided ideology but includes a fair amount of knowledge about the institutions of a democratic political system, which can be assumed in the liberal democracy of the inclusive state.

Regarding the influence of the model parameters, ceteris paribus, for higher values of parameter a, the "corridor" becomes smaller while for lower values, the equilibria collapse as the depreciation term dominates. In this case, only the origin is a stable equilibrium. This holds for fixed b, c, and d, and can be interpreted as follows: When the influence of the dynamics becomes stronger (larger a, especially relative to d), the capabilities of the groups change faster, driving the "corridor" towards the upper right corner. When, on the other hand, the dynamics becomes smaller (smaller a, especially relative to b), the influence of the other group increasingly dominates its own actions, and the relatively higher conflict parameter b drives the system towards the origin.

A higher value of the conflict parameter b increases the area of attraction of the origin, but an even higher value changes the stability of the high symmetric equilibrium. In this case there is only one stable manifold, as shown in Figure 5 for parameter values $a = 0.5$, $b = 0.2$, $c = 0.5$, and $d = 0.2$. The "corridor" has collapsed to an infinitely small width (actually, it has a dimension of one and is, in fact, a line), meaning that the desired equilibrium can only be reached if the initial values of x and y start exactly on the corridor and has thus measure zero. Otherwise, the system tends towards the boundary equilibria in the long run. That is, the corridor takes on the role as the separation manifold of the boundary equilibria (comparable to a weak Skiba manifold in an optimal control setup, as will be addressed in detail in Section 4). For even higher values, the positive equilibria collapse and only the one in the origin remains. The interpretation of this result is obvious: If the development of the capabilities of each group depends strongly and negatively on the other group's capabilities, no joint interest can develop, and no acceptable outcome can be expected as there will be a totally unproductive struggle for power without any possibility of gains from cooperation.

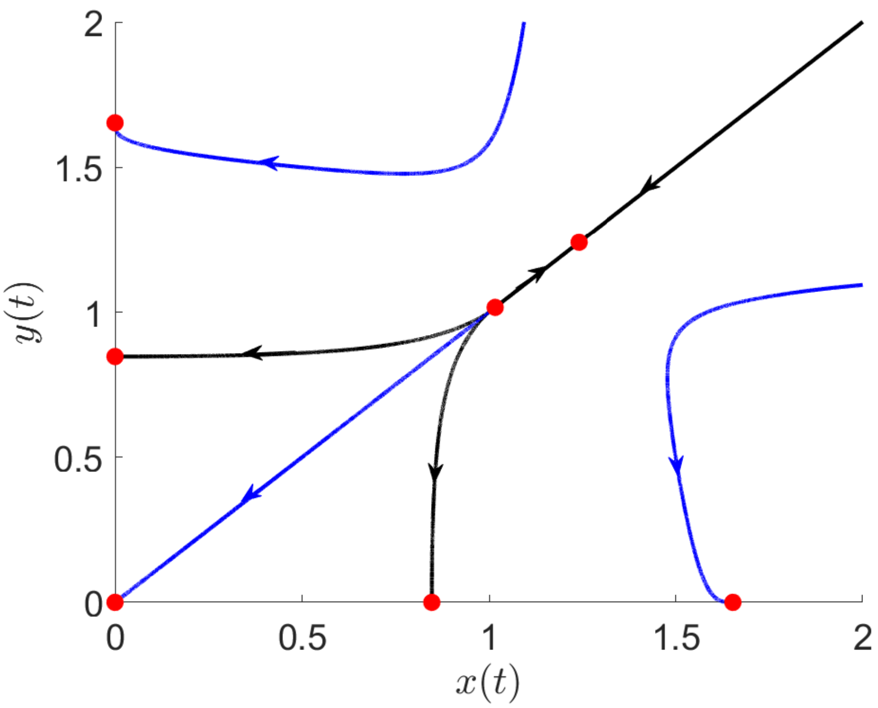


Figure 5: Phase diagram of system 3. Parameters: $a = 0.5$, $b = 0.2$, $c = 0.5$, $d = 0.2$.

Another interesting question relates to the extent to which the above sensitivities depend on the symmetry of the parameters and in particular to what happens if the conflict parameter of the elite b_x is different from that of civil society b_y. As we have assumed b to be relatively small so far, we now examine the case of a larger parameter $b_x = 0.1$ and $b_y = 0.01$, meaning that the elite has a (ten times) stronger negative influence on civil society's capacity than vice versa (a "predatory elite", so to say). The results are shown in the phase diagram in Figure 6.

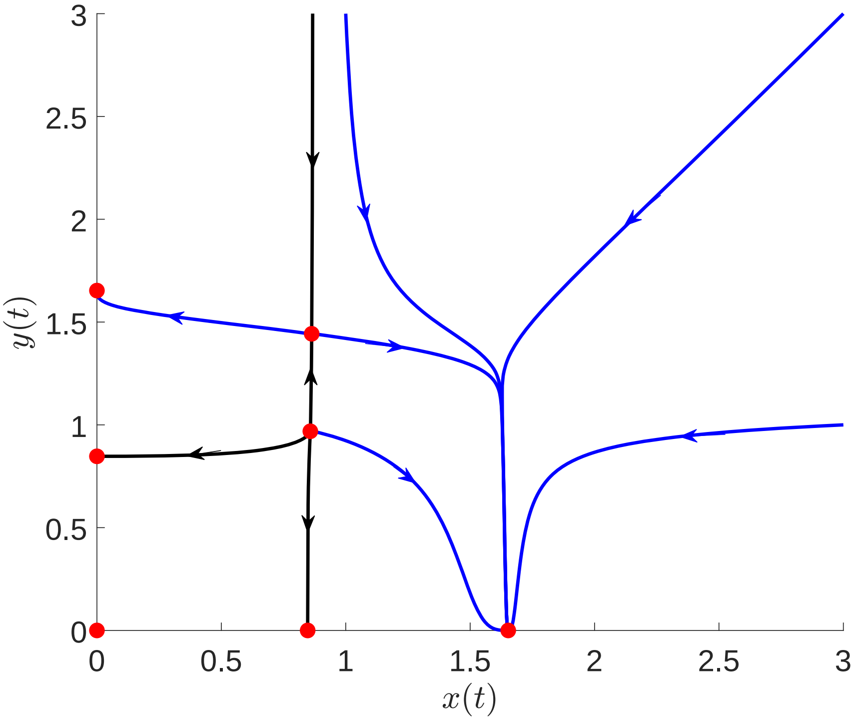


Figure 6: Phase diagram of system 3. Parameters: $a = 0.5$, $b_x = 0.1$, $b_y = 0.01$, $c = 1.4$, $d = 0.2$.

Comparing this diagram with Figure 5, we again see that we have an extremely narrow "corridor", albeit one that is now vertical. Otherwise, its qualitative behavior is analogous to that in Figure 5: Only the origin and two boundary equilibria on the axes are stable, and the system will converge to one of them. For larger b_x, the two unstable equilibria along the vertical line collapse, and we obtain the same qualitative behavior as without the corridor. For smaller b_x, a small but two-dimensional corridor with a stable equilibrium point arises, with a smaller region of attraction than for $b_x = b_y = 0.1$ (see, e.g., Figure 2). These results can be interpreted as follows: If only one of the groups has a higher potential to harm the other group, the possibility of a stable democracy will be smaller, ceteris paribus, than if their potential to harm each other is approximately equal. This strengthens the argument for the requirement of a low conflict parameter in favor of an inclusive state with the elite and civil society having equal strength. A strong civil society relative to the elite can lead more easily to anarchy while a strong elite (or government) relative to civil society can lead more easily to a dictatorship. The Soviet Union in its final stage may serve as an empirical example for the former, Russia under Putin as an example for the latter.

3.2 Bifurcation analysis

The conflict parameter b is key in our model as it governs the interaction between the elite and civil society, shaping the overall dynamics of the system. To study the role of this parameter in the dynamics of the model more systematically, we conduct a sensitivity analysis of the entire dynamical system. Mathematically, this involves applying bifurcation theory (see e.g., Kuznetsov [29]) to study the structural stability of the nonlinear dynamical system and to derive the values, i.e., the bifurcation curves at which the qualitative properties of the model dynamics change.[4] Figure 7 shows the bifurcation diagram in the conflict parameter (identifying b values where a structural change sets in) with the benchmark parameters $a = 0.5$, $b =$ bifurcation parameter, $c = 1.4$, $d = 0.2$. Going along the abscissa, the bifurcation parameter b is varied, and the corresponding equilibrium points of the state variable x are seen on the ordinate.

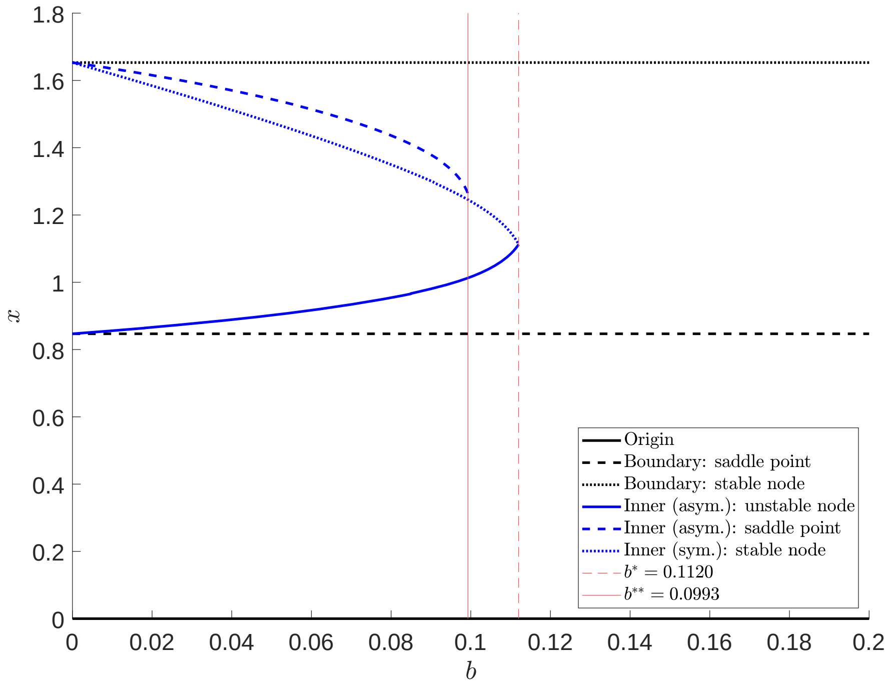
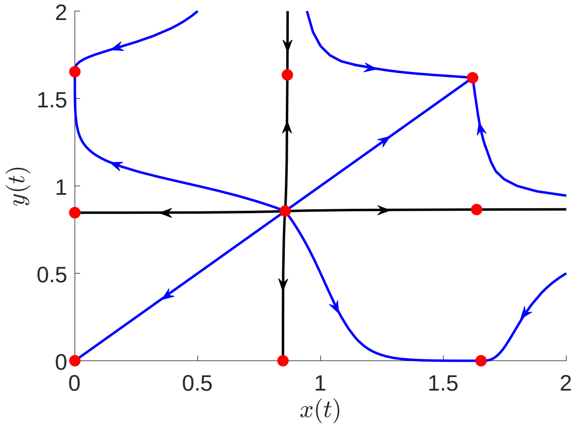


Figure 7: Bifurcation diagram for b with the benchmark parameters.

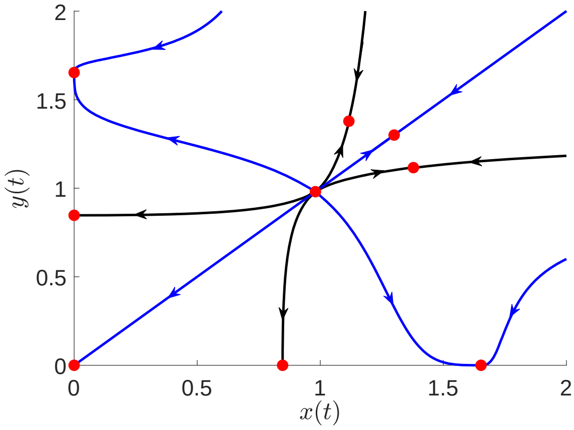
Starting from small values of $b < b^{**} = 0.0993$ (each group's capacity only affects the other's very weakly), we find that there are nine equilibria and four regions. For $b^{**} < b < b^* = 0.112$, there are seven equilibria and three regions with a heteroclinic connection (orbit) between the two interior equilibria. For large values of b with $b^* < b$, there are five equilibria and three regions.

Figure 8 shows the corresponding phase diagrams. Separatrices are shown in black and typical trajectories in blue. Note that every value of the parameter b in the bifurcation diagram in Figure 7 corresponds to a different dynamics (with corresponding phase portrait). In Figure 8 four cases are shown for increasing values of b (see Panels 8a-8d).

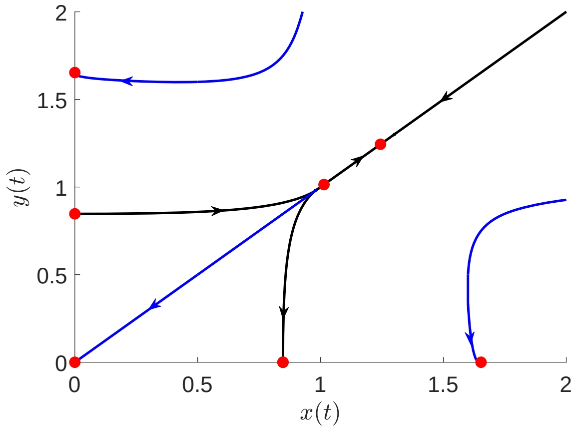
[4] Note that bifurcation theory is also used in Section 4 to study the qualitative stability of and structural changes of the optimal solution of the control model. However, a bifurcation analysis of an optimal control model is not directly related to that of a descriptive system and needs more sophisticated methods. For an overview, see Grass et al. [23] and Grass [22].



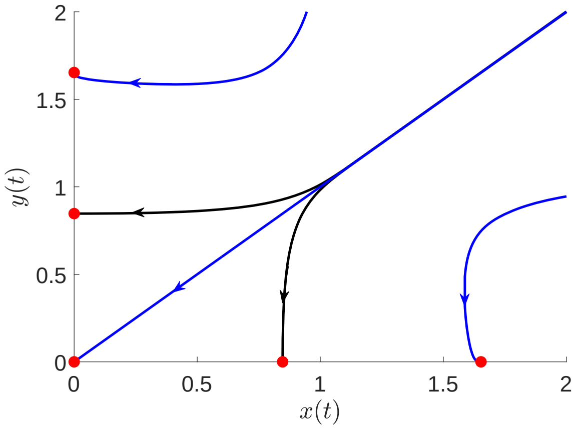
(a) Phase diagram for the benchmark parameters with $b < b^{**}$.



(b) Phase diagram for the benchmark parameters with $b = 0.09$.



(c) Phase diagram for the benchmark parameters with $b = b^{**} = 0.0993$.



(d) Phase diagram for the benchmark parameters with $b > b^{*}$.

Figure 8: Phase diagrams for the benchmark parameters with different values of b.

Panel 8a starts with the lowest area of the b values in Figure 7, i.e., with $b = 0.01 < b^{**}$. Here we find nine equilibria and the following four regions: In the lower left-hand region, every trajectory converges to the origin. In the lower right-hand and the upper left-hand regions, the trajectories converge to the boundary equilibria. Convergence to the stable interior equilibrium takes place in the upper right-hand region. This can be interpreted as follows: When there is a low level of negative interactions between the two groups, convergence to a liberal democracy is possible, provided that the initial capacities are sufficiently large. Wealthy liberal democracies can be stable if the difference in wealth between the two groups is not too big. If the capacities of the elite or civil society are very unequal, the conflict between them leads to anarchy or a dictatorship. If both groups are very weak (poor capacities), neither of them prevails but there is no output either – Hobbes's "state of nature". Attempts at escaping from this state are not possible if either the elite or civil society becomes stronger than the other, leading to an unproductive dictatorship or anarchy respectively. Only a social contract between the groups can lead to a productive outcome, as Hobbes conjectured; this does not require an absolute sovereign (who might become a tyrant) but can also be obtained by preventing too great an inequality between the two groups.

As b increases, the region with the interior equilibrium becomes smaller. For instance, for $b = 0.09$, we obtain the phase diagram in Panel 8b. The "corridor", albeit not narrow, is squeezed by the increase in the conflict parameter but is still an open region for higher capacities of a similar size for the elite and civil society.

The picture changes when b surpasses the threshold of $b = b^{**} = 0.0995$, where the two asymmetric interior equilibria collapse with the interior equilibrium. Now we have seven equilibria and three regions, as shown in Panel 8c. Again, we find convergence to the origin (Hobbes's "state of nature") for small capacities in both groups, convergence to two boundary equilibria (anarchy or dictatorship) for (initially) unequal capacities, and no convergence to the interior equilibrium, hence no stable liberal democracy. The upper interior equilibrium is a saddle point, the lower an unstable node, and the two are connected by a heteroclinic orbit, an unstable manifold. This can be interpreted to mean that along this orbit, with equal capacities of the two groups, there is an extremely small chance for democracy and it will be highly unstable. In the terminology of Acemoglu and Robinson, we can say that there is an extremely narrow corridor (of infinitely small width), which, if ever reached, will be of short duration. It exists for the special case of the conflict parameter b^{**} and equal capacity for both groups. Trajectories will leave this corridor under the smallest possible stochastic disturbance, and only unproductive equilibria will result.

For $b^{**} < b < b^{*}$, we obtain a similar phase diagram, with the "corridor" declining with increasing b until we reach b^{*}, where the two interior equilibria collapse and disappear. Here, and for even larger b, we have a phase diagram as in Panel 8d (plotted for $b = 0.12$). We are left with five equilibria and, again, with the three regions as before, all of them leading to unproductive long-run situations. The high conflict parameter prevents cooperation, and, depending on the initial distribution of the capacities of the two groups, the system results in anarchy (higher capacity of civil society), dictatorship (higher capacity of the elite), or the "state of nature" (low capacities for both groups). Note that the (now actually "narrow") corridor does not lead to a stable democracy but to Hobbes's "state of nature", which is a poor and unproductive stable situation. Clearly, this means that if the two groups interact strongly in a negative manner, a productive outcome is impossible, even if (and especially if) their capacities are not too different. At least one group is reduced to minimal capacity by the other, and if both are of similar strength, the result will be an equilibrium with the lowest possible living standard, even if the economy starts out from wealthy initial conditions.

4 Optimal control models

Next, we examine the possibility of an optimal policy for the elite's effort to manage its capability according to its own preferences. In this case, we assume that the elite, as the government, is active while civil society adapts passively to the actions of the government. This is the case in a not well-organized civil society that may, for example, consist of various groups without a common interest vis-à-vis a well-organized selfish elite. Another possibility would be a society with an established corporate order, such as the medieval estate system, also held together by an uncontested ideology enforced by the Catholic church. The elite forms the government with strong instruments of power, including loyal civil servants, judges, military, police, and hangmen, to pursue its goals as best they can without considering the preferences of civil society. In principle, the reverse problem of an optimizing civil society with a passive elite could also be analyzed, but this case is historically much less relevant; moreover, it would produce analogous results as below by simply exchanging the x and y coordinates due to the symmetric nature of the problem under consideration. For the alternative of a benevolent government (elite), see Section 5 below.

We assume an objective function containing the share of the elite in the output of the society and the costs (assumed to be quadratic) of the elite's actions $u(t)$ as arguments. Output is determined by the Cobb-Douglas production function and is positive as long as both x and y are positive. Civil society adapts passively (i.e., $v(t) = 1\ \forall t$). As in the previous section, we consider only the symmetric case. The discount (time preference) rate is denoted by $r > 0$, which encapsulates a farsighted decision maker with a small value and a shortsighted one with a large one.

The optimal control problem of the government (the elite) as the decision maker is:

$$\max_{u(t)} \quad \int_0^\infty e^{-rt} \left(x^\alpha y^{1-\alpha} \frac{x}{x+y} - \beta u^2 \right) \mathrm{d}t \tag{4}$$

subject to

$$\dot{x} = a\frac{x^2}{c+x^2}\left(1 - b\frac{y^2}{c+y^2}\right) u - dx, \tag{5}$$

$$\dot{y} = a\frac{y^2}{c+y^2}\left(1 - b\frac{x^2}{c+x^2}\right) - dy. \tag{6}$$

This standard optimal control problem is solved by application of the Maximum Principle (for the optimality condition see Appendix A); the calculations of the bifurcation diagrams require sophisticated numerical method such as the continuation of boundary value problems (see Grass et al. [23], chapter 7).

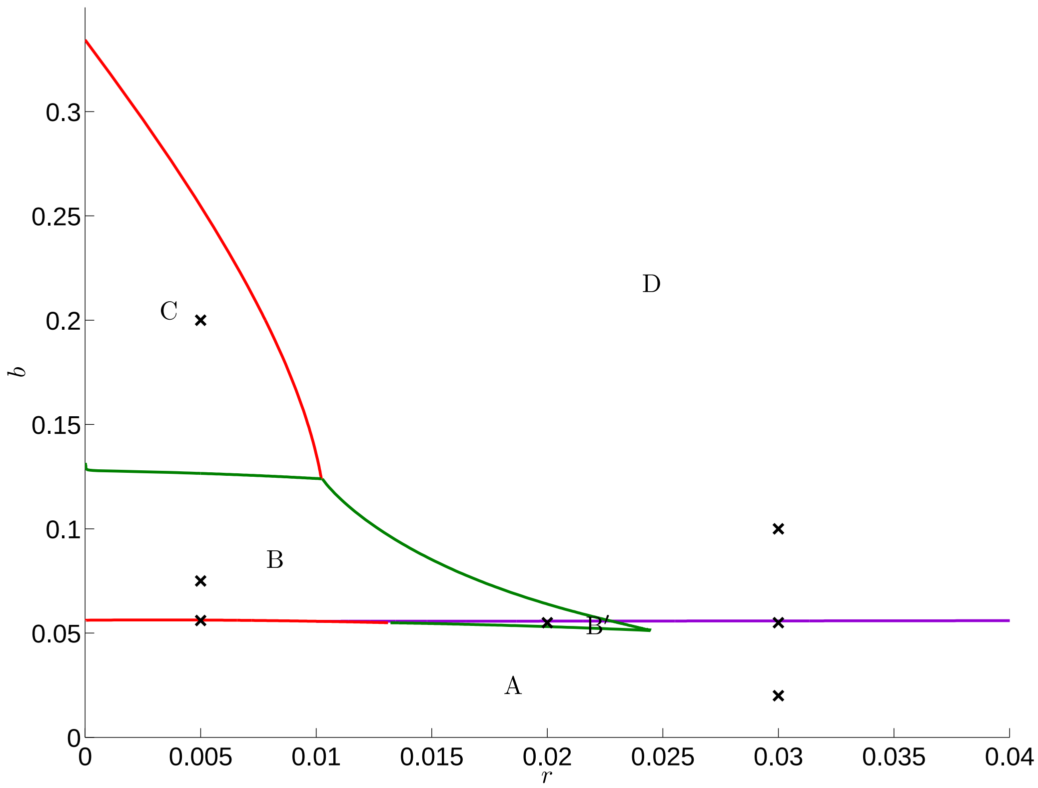


Figure 9: Bifurcation diagram of the optimal control problem in the (r, b)-space.

A bifurcation analysis of this optimal control problem is most interesting for the rate of discount r and the conflict parameter b as bifurcation parameters. Its results are shown in Figure 9. There are four regions A, B, C, and D with qualitatively different behavior in terms of the resulting equilibria and the trajectories leading to them. In the following phase diagrams, optimal equilibrium points are denoted by blue dots, triple Skiba points by red dots, Skiba curves by black curves, weak Skiba curves (leading to an unstable equilibrium) by dashed blue curves, and paths with zero values of the control by green curves.

The bifurcation and phase diagrams reveal that the optimal control model exhibits a history-dependent solution structure over a wide range of parameter settings. History or path dependence implies that the long-run optimal solution is contingent on the systems history, i.e., on the initial conditions (x_0, y_0). A (weak) Skiba curve delineates regions within the phase diagram where the optimal solutions converge to different long-run steady states. When the initial conditions lie precisely on a Skiba curve, the system exhibits indifference, allowing for two equally optimal trajectories. These trajectories may either converge to different steady states or to the same one via divergent paths. However, if the initial state lies on a weak Skiba curve, the system does not show indifference. Instead, the trajectory converges uniquely to a specific steady state that is accessible only when originating from this curve. Adjacent regions separated by the Skiba curve lead to distinctly different optimal solutions. In cases where two Skiba curves intersect, a triple Skiba point may emerge. At such points, the decision maker is indifferent among three equally optimal solutions, each corresponding to a different trajectory emanating from the intersecting Skiba curves. For details, see Grass et al. [23] (chap. 5).

In particular, we get the following results:

- Region A: Here the conflict parameter is small ($b < 0.056$), meaning that each group has a small external effect on the other group's capacity, which may be due to appropriate institutions enhancing cooperation. A phase portrait for $r = 0.03$ and $b = 0.02$ is given in Figure 10. It shows that there are two locally stable optimal equilibria, the origin and an interior equilibrium with positive values for both x and y. The black curve is a Skiba curve, separating regions 2 and 3 above an unstable equilibrium on the y-axis. In region 1, all trajectories converge to the origin. Here the elite, starting from 0, first applies decreasing efforts to produce a positive output together with civil society. The higher the initial capacity of the civil society, the longer this policy lasts. However, at some point the elite realizes that the cost of its efforts does not yield a sufficiently high share of the output in its favor and it reduces its efforts, thereby diminishing output as well as civil society's and its own capacities until the "Behemoth" or poverty line equilibrium of the origin is reached. This happens under a relatively low initial capacity of the society. This region corresponds to region III in Figure 2.

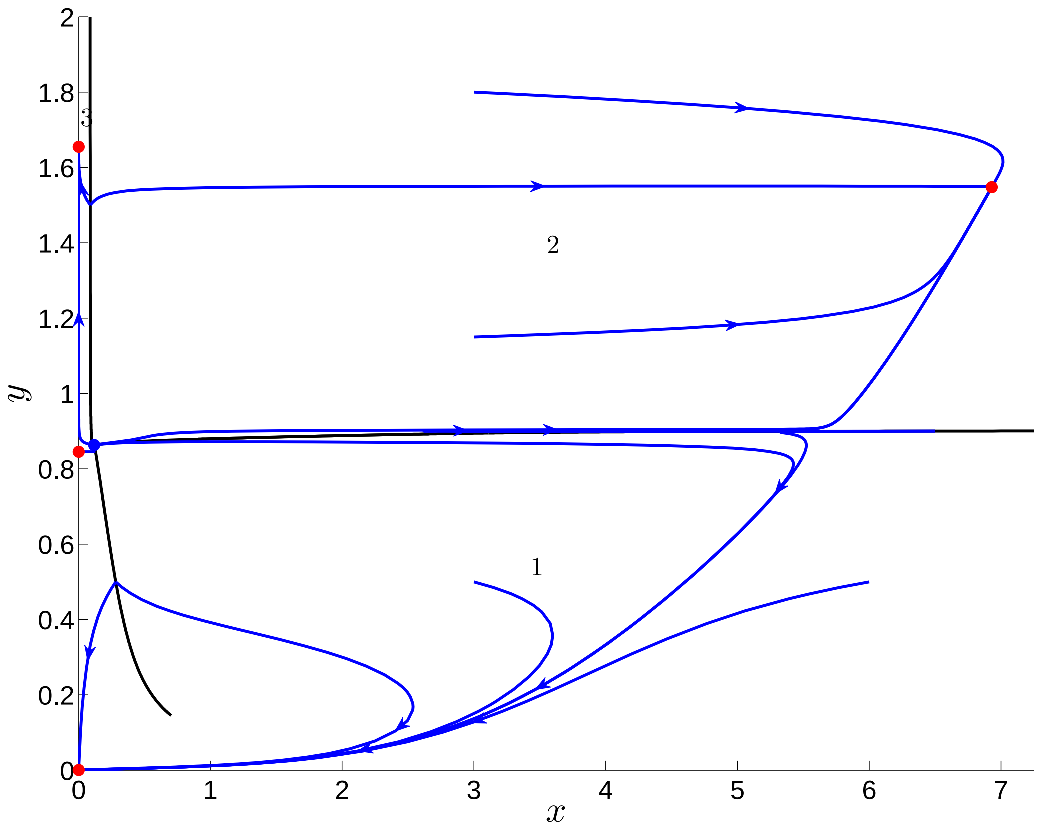


Figure 10: Phase portrait region A, $b = 0.02$, $r = 0.03$.

Region 2 is the case where a productive outcome is obtained, namely the case of a stable liberal democracy. On and above the horizontally oriented Skiba curve (the black line), all trajectories converge to the interior (locally stable) equilibrium, with both groups cooperating to obtain a positive output. To arrive at this solution, the elite applies a positive effort, increasing it when the capacity of civil society is relatively small and decreasing it when civil society is stronger. The aim of the elite is to ensure a sufficiently high capacity of civil society through higher output in the former case and a sufficiently small capacity of civil society to reach the elite's goal of gaining a maximum share of the output in the latter case. In both cases, the same equilibrium is reached. This corresponds to region I in the uncontrolled case in Section 3.

The narrow region 3 (left of the vertically oriented Skiba curve) leads to an equilibrium without the elite but with a positive civil society, the case of anarchy as in region IIb (Figure 2) in the uncontrolled case. This equilibrium is reached from high initial values of civil society's capacity but very small initial values of the elite's capacity. Although this may seem a better solution than the "Leviathan" one, it is nevertheless unproductive: No output is produced. We may interpret this to mean that a minimum capacity on the part of the elite (a minimal state) is required to give it some incentive to make the effort to drive the system towards a liberal democracy (into region 2). In a comparison of different societies, this illustrates the historical fact that liberal democracies can only develop in societies with a mature institutional environment, where both the elite and civil society already have sufficient capacity and a certain level of prosperity (output) when a nascent democracy starts developing. The fact that the rise of democratic political systems did not take place before the modern age concurs with this result of our model.

It should be noted that this scenario holds irrespective of the value of the discount rate r, as can be seen from the bifurcation diagram in Figure 9: There is no bifurcation within region A. Of course, the position of the equilibrium point and the shape of the trajectories will differ for different values of r but the qualitative behavior is the same. An example of such a scenario is shown in Figure 11 for the same rate of discount and a larger conflict parameter $b = 0.055$. Here we have a weak Skiba curve (the horizontally oriented dashed curve) separating regions 1 and 2, with an unstable equilibrium point near the interior equilibrium. The interpretation is the same as for Figure 10. A corresponding figure with slighly increased conflict parameter ($b = 0.056$) can be found in the Appendix C (see Figure 20).

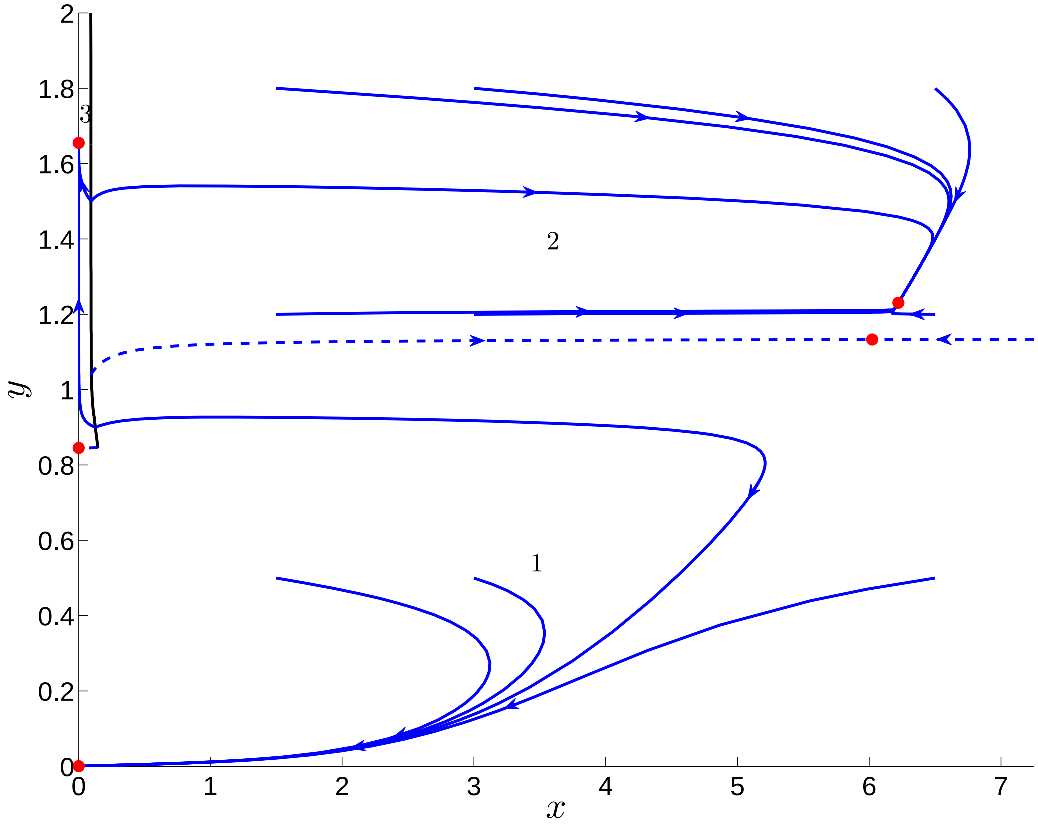


Figure 11: Phase portrait region A, $b = 0.055$, $r = 0.03$.

- Region B is characterized by higher values of the conflict parameter and low values of the discount rate (see the bifurcation diagram in Figure 9). Here we find a qualitatively different behavior of the system, where the interior equilibrium of region A is replaced by a (locally stable) limit cycle. Figure 12 shows the phase diagram for the parameters $b = 0.075$, $r = 0.005$, which corresponds to the cross in region B in the bifurcation diagram. Region 1 in this diagram is like region A in the bifurcation diagram, but in region 2 we find (for initial values of the elite's capacity beyond a minimal one and a sufficiently high capacity of civil society) a limit cycle instead of an interior equilibrium point. The scenario of region B occurs with the elite's high preference for long-term results, which may be interpreted to substitute to some extent for the higher conflict parameters yielding a liberal democracy, although it is characterized in this case by regular oscillations of both the elite and civil society.

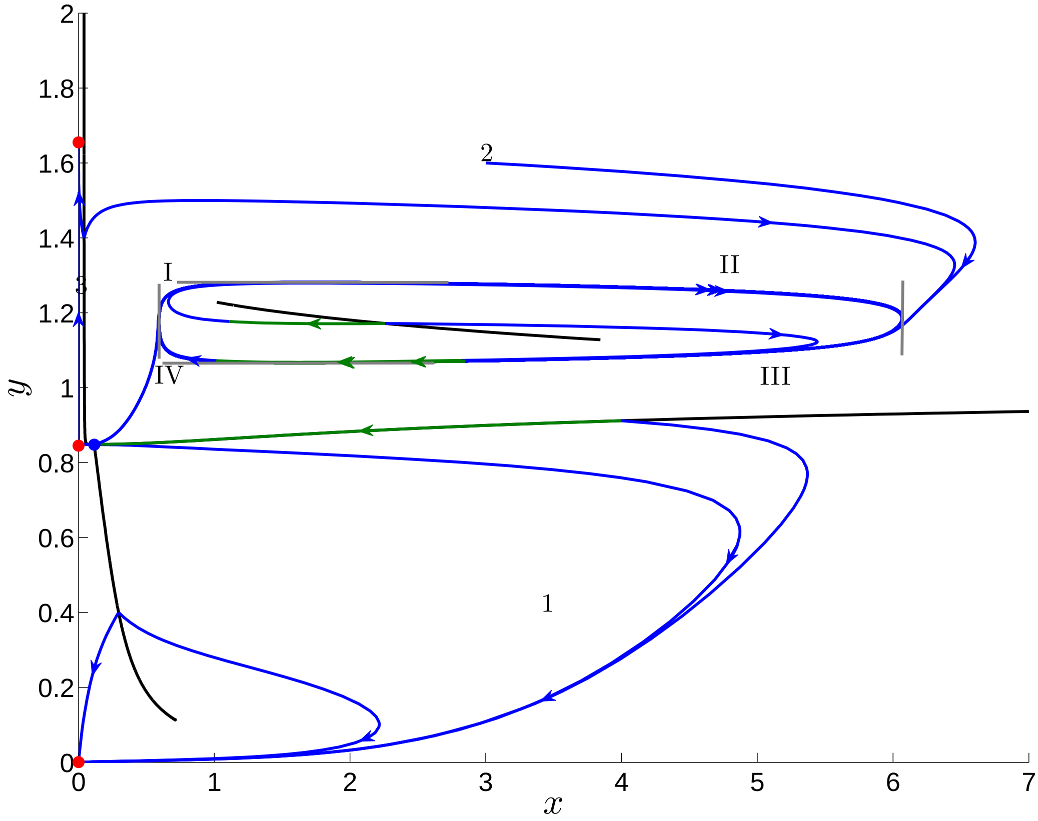


Figure 12: Phase portrait region B, $b = 0.075$, $r = 0.005$.

The limit cycle is illustrated by the trajectories of the elite's effort (u) and capability (x) and civil society's capability (y) over time in Figure 13. It can be interpreted as follows: The elite starts out from the vertically oriented Skiba curve or from the triple Skiba point in Figure 13 by applying some effort to allow for positive production by enhancing the capacity of civil society (phase I in Figure 13). In the upper horizontally oriented branch of the cycle in Figure 12, it applies positive but decreasing effort to hold the capacity of the civil society approximately constant (phase II in Figure 13). When the costs of the elite's efforts become larger than the share of the output it can extract, it puts more emphasis on the cost part of its objective function by reducing its effort to zero (phase III). In doing so, the capacity of civil society decreases slowly (the lower horizontally oriented branch of the cycle in Figure 12) and the elite exploits civil society (the government degenerates into a body extracting rents from the private sector) until the output becomes so small that the elite cannot obtain more resources than the cost of a new effort. When the capacity of civil society decreases to a minimum, the elite renews its efforts, allowing the capacity of civil society to rise (phase IV in Figure 13), and the cycle starts again.

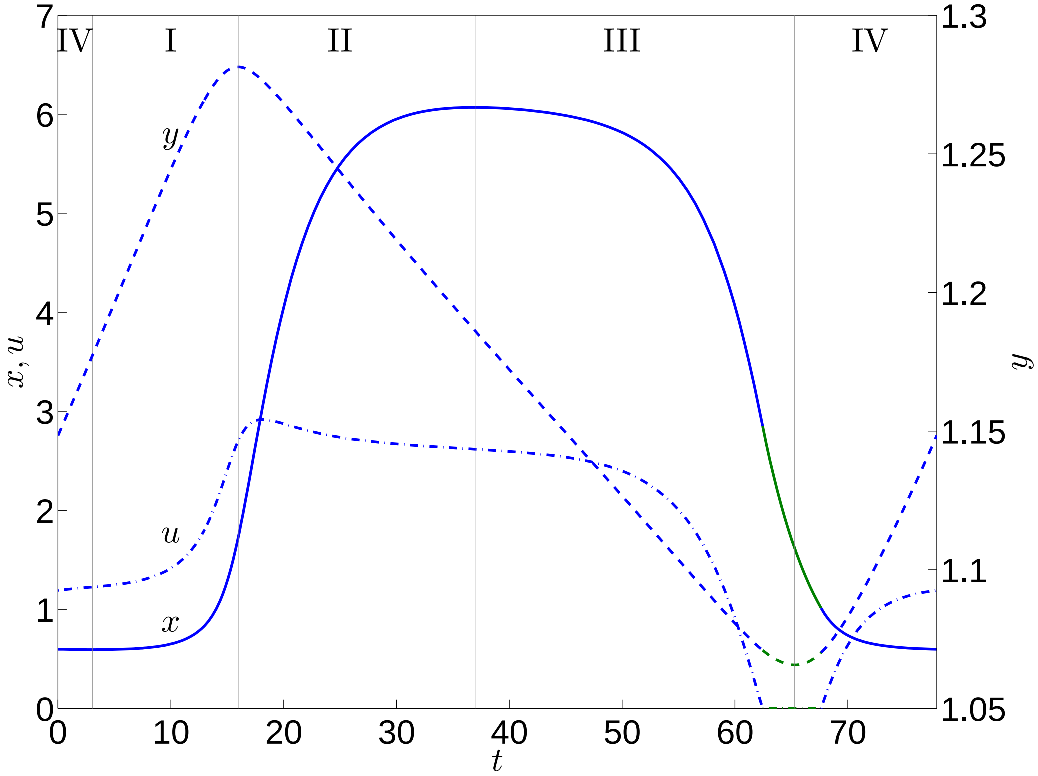


Figure 13: Time paths of the elite's effort (u) and capacity (x) and civil society's capacity (y) in the cycle.

The interesting point in this case is the presence of a political business cycle, without requiring exogenous elements such as election dates as in Nordhaus [31] type cycles or alternating political parties as in Hibbs [24]-Alesina [7] partisan cycles (although the latter may be part of the picture in a two-party democracy). The cycle results endogenously from the optimal policies of the elite under the given assumptions about the dynamic system and the parameters for this case. It should be noted, however, that the elite in region 2 also has the option of going for an equilibrium on the y-axis (left of the vertically oriented Skiba curve), resulting in a society without government, or going into region 1 (from the horizontally oriented Skiba curve), resulting in a society at the poverty line. Both of these options give the elite the same value of the objective function as in the cyclical solution.

- Region B': This small region is an interesting special case of region B that extends into region A. Here we also find a limit cycle; see Figure 14. Regions 1, 3, and 4 correspond to regions 1, 2, and 3 respectively in the phase diagrams for regions A and C. Now there is a narrow region 2, however, which contains the limit cycle. It lies between a weak Skiba curve and the strong Skiba curve separating regions 1 and 2. The Skiba point near the y-axis and the horizontally oriented strong Skiba curve play the same role as in region B as starting points of the limit cycle. The locally unstable equilibrium on the weak Skiba curve drives the dynamics of the system down and back towards the y-axis, thereby creating the limit cycle within region 2. Region 2 in region B' corresponds to region 2 in region B, but in contrast to the dynamics in region B, we still have the stable interior equilibrium in region 3 (as in region 2 in region A). Region B' can be seen as the intermediate region between regions A and B in the bifurcation diagram. The interpretation of the cycle is analogous to that in region B. If we move upwards in this phase diagram (to higher initial values of civil society), the cycle becomes narrower and vanishes when the weak and the strong Skiba curves collapse, the system bifurcates into region B, and the interior equilibrium point vanishes.

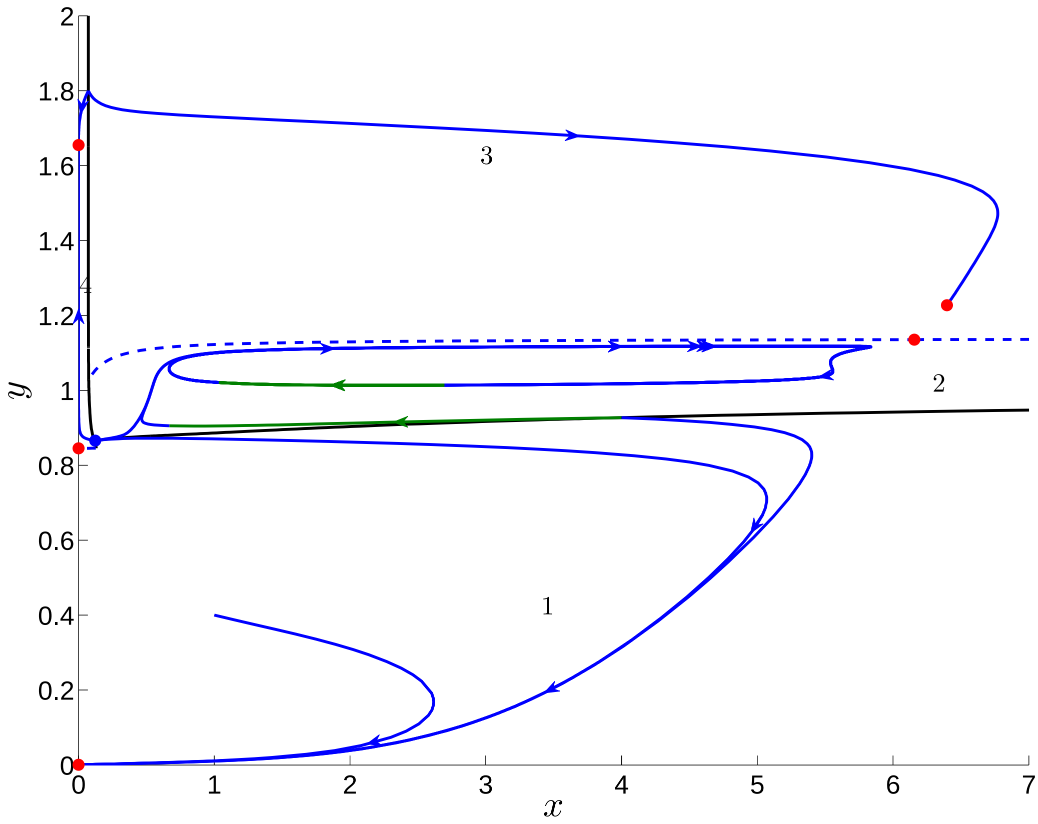


Figure 14: Phase portrait region B', $b = 0.055$, $r = 0.02$.

- Region C: In this region of the bifurcation diagram (Figure 9), there are even higher values of the conflict parameter b. As the right-hand boundary of region C shows, with increasing values of b, the discount rate has to be smaller to remain in C. Thus, as in region B, there is a trade-off between higher negative externalities and a higher long-term orientation of the elite towards staying within region C. Figure 15 presents the phase diagram for $b = 0.2$, $r = 0.005$. It shows that the cycle in region B is no longer present, but there is a locally stable interior equilibrium instead, with a lower value of x than in region A. By being inactive, the elite can again drive the system to a boundary equilibrium at the y-axis along the paths of the narrow region 3. Alternatively (with the same value of the objective function), by making a little effort, it can reach the interior equilibrium in region 2. From initial situations of positive values of x and y to the right of the vertically oriented Skiba curve in region 2, the elite can reach this desirable equilibrium (from a democratic point of view) by first reducing its positive effort. When the costs of the effort become larger than the utility of the elite's share in output, it reduces its effort to zero and, with a final push, drives the system into the interior equilibrium. The declining (quasi-horizontal) Skiba curve is the border between regions 1 and 2. For initial situations below that curve, the elite will always go to the origin (the poverty line society), either directly or by first applying effort and then reducing it towards zero.

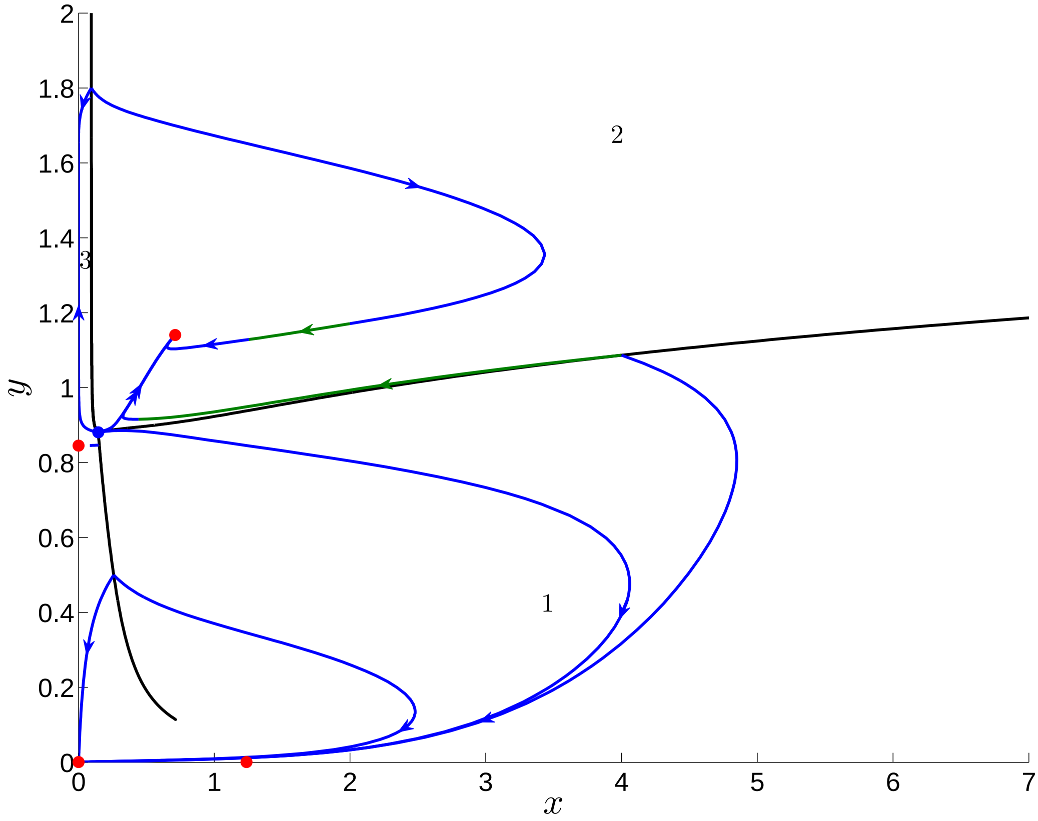


Figure 15: Phase portrait region C, $b = 0.2$, $r = 0.005$.

- Region D: This region covers all cases with higher conflict parameter b than in region A and higher discount rates than in regions B and C. It characterizes a society with not very inclusive institutions (high political externalities) and a relatively myopic elite. In this case, no productive interior equilibrium is possible, and the elite will always decrease its effort to zero. A phase diagram for $b = 0.1$, $r = 0.03$ is shown in Figure 16. To the left of the (only) Skiba curve, the elite may drive the system to a boundary equilibrium with positive y; hence civil society may continue to exist if its initial capacity is sufficiently high. Otherwise, and with the same optimal value of the elite's objective function, it will lead the dynamic system to the origin, the poverty line equilibrium, either directly by reducing its effort or, for relatively low initial values of x, via the detour of first applying decreasing effort and then reducing it towards zero. In any case, no output is produced in the equilibrium. The discussion of a corresponding case with increased conflict parameter $b = 1$ is relegated to the Appendix C (see Figure 21).

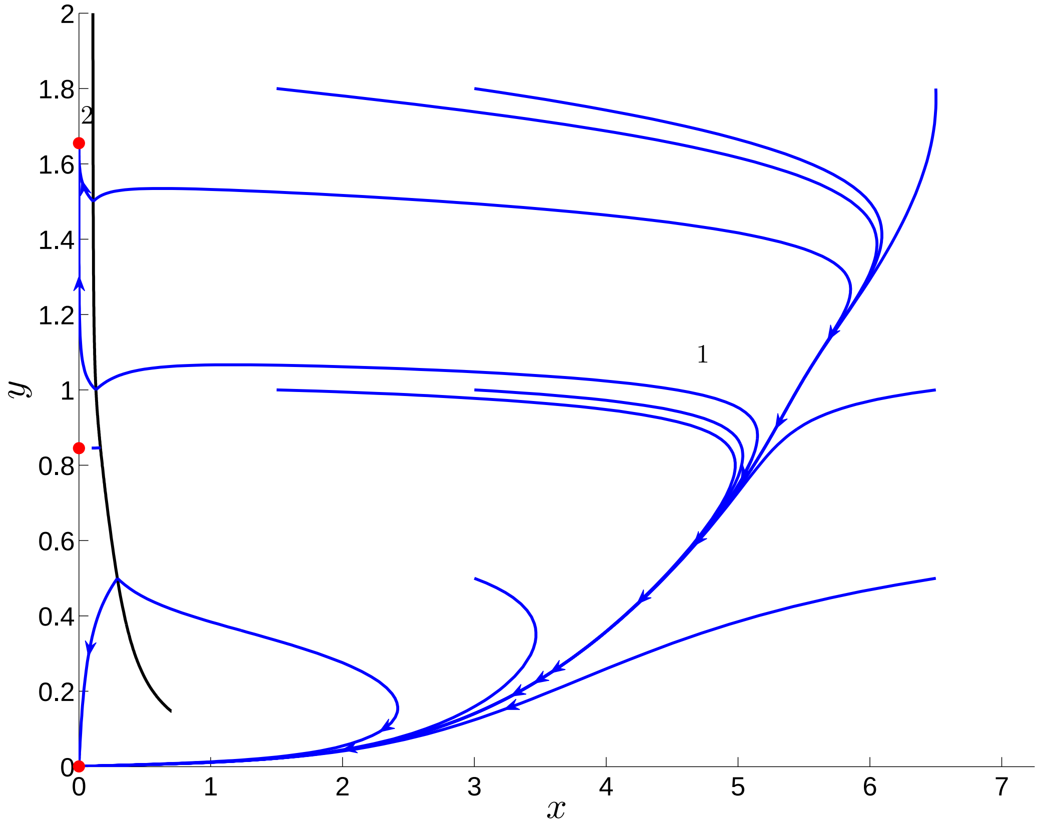


Figure 16: Phase portrait region D, $b = 0.1$, $r = 0.03$.

To conclude the dynamic analysis for the optimizing elite (or the government), we find a variety of dynamic behavior depending on the size of the parameters b and r and the same long-run outcomes as in the uncontrolled scenarios, but in addition, there is a (political) limit cycle instead of a stable equilibrium for intermediate values of b (intermediate political externalities) and low values of r (long-term orientation of government policies). Under these circumstances, a liberal democracy will be stable but oscillating between rising and falling capacities of the elite and civil society; these oscillations are not synchronized but driven by oscillating government policies. However, the analysis of Skiba points and curves shows that there are situations where the elite may be indifferent regarding an unproductive and a productive outcome, which introduces an element of political instability into the political system over time, which is similar but qualitatively different from the narrow corridor in the work of Acemoglu and Robinson.

5 A dynamic game

In this section, we extend the examination of our model to a dynamic game. Here, both the elite and civil society act rationally in accordance with their own optimal decisions, anticipating strategic interaction with the opposing party, referred to as players in a game-theoretical context. In this respect, differential game theory – while incorporating various mathematical distinctions and generalizations (see, e.g., Dockner et al. [17], Basar and Zaccour [11]) – can be seen as an extension of the optimal control approach, which will therefore serve as the appropriate solution tool. We first start with a noncooperative game, where the players optimize their objective functions individually without a binding agreement to cooperate, and then turn to a cooperative game, where the two groups act jointly to optimize an objective function which is a linear combination of their objective functions.

5.1 Noncooperative open-loop Nash equilibrium solutions

With noncooperative open-loop Nash equilibrium solutions, both players, the elite and civil society, optimize their own objective function in a noncooperative way along time, which reads:

$$\text{Elite:} \quad \max_{u(\cdot)} \quad \int_0^\infty \mathrm{e}^{-rt} \left(\frac{x\left(x^\alpha y^{1-\alpha}\right)}{x+y} - \beta u^2 \right) \mathrm{d}t \tag{7a}$$

$$\text{Civil society:} \quad \max_{v(\cdot)} \quad \int_0^\infty \mathrm{e}^{-rt} \left(\frac{y\left(x^\alpha y^{1-\alpha}\right)}{x+y} - \beta v^2 \right) \mathrm{d}t, \tag{7b}$$

subject to the dynamics (1) and (2). Each player optimizes their own share of total production, reduced by the convex cost of efforts to enhance their own capacity. This mirrors the optimal control solution from the previous section, with the distinction that here, $v(t)$ is also optimized, whereas in the prior analysis, $v(t)$ was exogenously fixed at 1. For the game-theoretic solution, we assume symmetry between both players, full information, and simultaneous decision making. This implies the absence of a hierarchical (Stackelberg) structure, making the Nash equilibrium the appropriate game-theoretic framework. We employ an open-loop commitment structure for several reasons, as outlined below.

A Nash equilibrium can be implemented with several commitment structures, the most common being open-loop (OL), closed-loop (CL), and feedback (FB, also known as Markovian). Typically favored by economists for its strong time consistency (subgame perfection), the feedback structure is accompanied by considerable challenges. In the context of the present model, applying the FB solution is quite complex and prevents comparability with the optimal control framework. The FB solution relies on the Hamilton-Jacobi-Bellman principle, diverging from the Maximum Principle approach used in optimal control, OL, and CL solutions of differential games. Consequently, qualitative analyses – such as those represented by bifurcation diagrams and phase portraits (as discussed extensively in the previous sections) – are rarely developed in the literature, with a very few exceptions based on highly stylized assumptions. Moreover, the FB Nash solution can yield infinitely many potential outcomes, complicating the transition from a singular decision maker solution (as in Section 4) to a noncooperative strategic one by two players (the elite and civil society). In cases where the parameter space allows for Skiba-type or cyclical solutions, selecting one solution would raise the question as to why another, equally viable, solution was not chosen. Restricting the admissible set of FB Nash solutions to counter this issue undermines the validity of qualitative analyses.

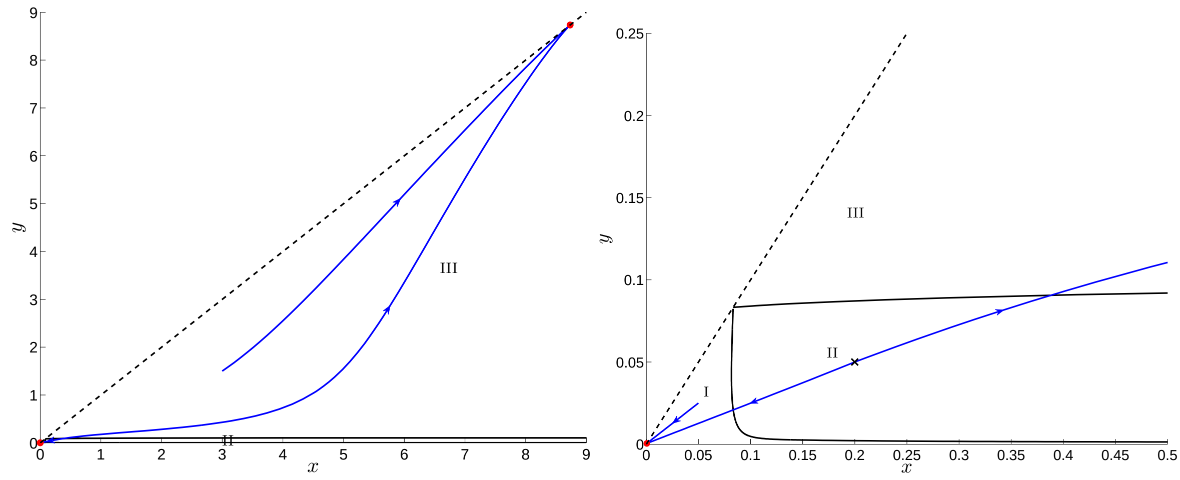
While the CL solution, based on the Maximum Principle, facilitates cross-comparisons with previous findings, it is subject to similar criticisms concerning solution multiplicity. The standard method of using first-order conditions to derive CL-feedback control functions for the canonical system produces a single solution, despite the possibility of many more (this is the informational non-uniqueness property discussed, for example, by

Basar and Olsder [10]). The OL solution, in contrast, avoids these particular issues, although it introduces other complexities due to the intricate structure of the model, which tends to promote Skiba-type behavior. These theoretical challenges remain largely unexplored in the current literature on differential games. A more detailed discussion will follow in conjunction with the presentation of our results utilizing the OL Nash solution. The optimality conditions are formulated in Appendix B.

The assumption of self-commitment by both players over the entire (infinite) time horizon to the strategies chosen at the initial point of time, which is constitutive to the OL Nash equilibrium solution concept, can be justified in the present context by the following considerations. First, this equilibrium solution is weakly time consistent (see Basar [9]), meaning that it will be followed by both agents, also under re-optimization along the time path for all t. As we assume a deterministic framework without stochastic or other exogenous shocks, the lack of strong time consistency (Markov perfection) is less of a problem than in a stochastic framework. At least the OL Nash equilibrium solution may serve for comparison with other (especially the one-sided optimization solution) scenarios for the same model. Second, self-commitment may also be interpreted as arising naturally under a given institutional environment (either authoritarian or democratic) which is shaped according to rules that are being followed by the groups acting within that environment out of their rational commitment to these rules in view of alternative ones, which may seem less attractive to the players. Hence, in the absence of exogenous shocks, there may be no incentive to deviate from the OL Nash equilibrium path by either group.

In contrast to the optimal control analysis in Section 4, where parameters r and b were varied to characterize qualitative solution behaviors within a bifurcation diagram, we adopt another approach here. While such an analysis is, indeed, a challenging effort – one that, as previously noted, has rarely been explored in the differential game literature so far – our primary focus lies on a different, yet equally significant, result illustrated by selecting two specific values of the conflict parameter while maintaining a constant discount rate. The findings show critical implications for the solution to the model and potential coordination efforts (negotiations or institutional requirements) between the elite and civil society.

In the analysis depicted in Figure 17, we hold the conflict parameter and the discount rate constant at $b = 0.2$ and $r = 0.03$ respectively while varying the initial conditions for x and y. Figure 19a illustrates the dynamics for this setting while Figure 17b zooms in on Figure 17a to show the results for small values of x and y. The phase diagram reveals two steady states: a high, symmetric interior equilibrium and the origin, with no asymmetric steady state present. The dashed black line depicts the 45-degree line (the symmetry of the problem implies analogous behavior to this scenario but with the opposite roles of the players), and, as in Section 4, the black lines indicate qualitative changes in the solution structure. However, unlike the Skiba curves in the optimal control solution, these lines delineate regions within the phase space where the players' preferences for the steady state coincide or diverge. Diverging, or in other words opposite, preferences, in this regard, refers to a situation where the elite and civil society each favoring a different steady state.



(a) Phase portrait.

(b) Zoom of the phase portrait.

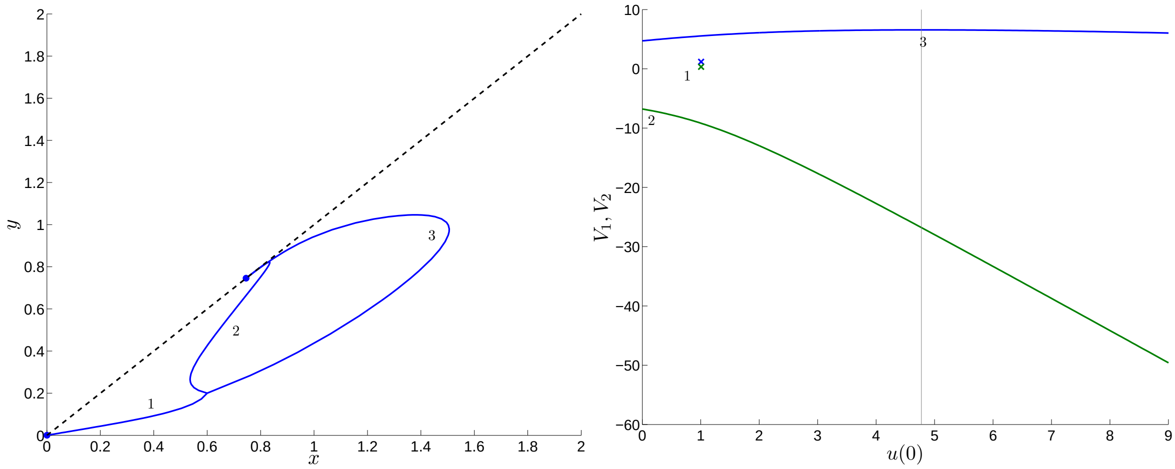
Figure 17: OL Nash solution for $b = 0.2$ and $r = 0.03$. The cross denotes a specific initial constellation in the opposite preference region (labeled as region II), where both players have the preference to converge to different equilibria.

The intuition is as follows: For initial states in region I (low initial capabilities of both groups), both players find it optimal to converge to the origin, following a uniquely determined trajectory in the state space. For the point $(0.05, 0.025)$ in region I in Figure 17b, for instance, the costs of trying to overcome the situation near the poverty line make it advantageous to stay at or move directly into this equilibrium. The value functions (values of the objective functions) of the players at this point are $(V_1, V_2) = (0.08, 0.04)$ for the origin and $(-75.8, -429.4)$ for the interior equilibrium; hence, both are, by far, better off when moving into the origin. Similarly, initial states in region III result in a mutual preference for the high steady state, also along a unique path. This means that a noncooperative equilibrium strategy under appropriate institutional circumstances (commitment) and without exogenous influences will lead to a balanced (here: equal capabilities for the elite and civil society) steady state, which in this model is a stable liberal democracy.

However, for initial conditions in region II, a conflict arises: Both players disagree on the preferred steady state. For example, taking an initial point $(x_0, y_0) = (0.2, 0.05)$, marked by a black cross in region II in Figure 17b, meaning the elite begins in a stronger position relative to civil society, the value functions of each player, $(V_1, V_2) = (0.31, 0.08)$ for the origin and $(41.75, -86.73)$ for the high steady state, reflect different and, more precisely, opposing preferences, with the elite favoring the path toward the high steady state and civil society preferring convergence to the origin. This scenario yields two distinct Nash equilibria – each leading to a different steady state – yet each player would ideally commit to the equilibrium path that maximizes their own objective functions. In essence, this setup in place for any initial point within this region mirrors the convex-concave nature of the dynamics responsible for the emergence of Skiba curves in the optimal control model, although here the differential game context modifies this property. Instead of a Skiba threshold, the model presents a long-term outcome in which each trajectory implies a distinct value function. To account for this fact, we will refer to this effect by *opposite preference* point or region, respectively. To our knowledge, unlike the – albeit – sparsely addressed Skiba points (see, e.g., Dockner and Wagener [18] or Dawid et al. [15]), this phenomenon in differential games has not been explored in the literature at all.

The result of this scenario can be interpreted as follows: In regions I and III, we have a unique OL Nash equilibrium, with the poverty line and liberal democracy outcomes respectively. For low initial capabilities of both groups, the poverty line society will be the outcome. As long as both groups are weak (small in number or poor, for instance), Hobbes's "state of nature" may last for a long time. A stable liberal democracy, on the other hand, may result when the initial capabilities of both groups (and especially of civil society) are relatively high. This is similar to the result for region A (in Figure 9) in the one-sided optimization environment described in Section 4. Both when civil society follows its own preferences (including a desire for a high share in production) and behaves strategically or when it adapts passively to the policy of the government (the elite), a stable liberal democracy can be established. In regions B, B', and C in the optimal control case (Figure 9), this outcome of a liberal democracy is also possible (in regions B and B' characterized by cyclical behavior) but it will not be stable due to the option of the elite to drive the system to a boundary equilibrium. Similarly, in the OL Nash game, the outcome of trajectories starting from unequal initial capabilities in region II is also an unstable liberal democracy, which, here, is due to the non-uniqueness of the OL Nash equilibria going either to the poverty line society or to the liberal democracy. Both in the optimum control and the OL Nash game framework, the possibility of establishing and maintaining a liberal democracy in this situation will depend on institutional measures (for instance, a democratic constitution established by a contract between the two groups) to favor the productive outcome of the liberal democracy.

Figure 18 illustrates the dynamics under a high conflict parameter, $b = 1$, with the discount rate held constant. As both players of the dynamic game are symmetric the phase diagram is mirrored at the 45-line. Unlike previous diagrams, we do not demarcate distinct regions in the phase plane with black lines but highlight a different phenomenon instead. Panel 18a displays a phase diagram for this game while Panel 18b shows the associated objective value functions for both players (elite: blue; civil society: green) corresponding to a specific initial point, namely $(x_0, y_0) = (0.6, 0.2)$. Notably, the game now diverges from the optimal control framework discussed in Section 4. In optimal control models, a unique trajectory typically emerges as the optimal solution when initial conditions lie outside the Skiba surface; alternatively, two or three optimal paths may exist if initial conditions are on the Skiba surface or there is a triple Skiba point. In contrast, within the OL Nash equilibrium solution, a continuum of potential trajectories may arise, none of which is equally preferable for both players.



(a) Phase portrait. (b) Objective value functions.

Figure 18: OL Nash solution for $(x_0, y_0) = (0.6, 0.2)$, $b = 1$, $r = 0.03$.

Panel 18a reveals two distinct steady states: (i) the origin, where $x = y = 0$, representing a scenario in which both players hold no power and no production occurs, and (ii) a symmetric inner steady state, where $x = y > 0$, signifying a balance of power between the players, with both producing and sharing a high output. If the elite and civil society, for example, start at $(x_0, y_0) = (0.6, 0.2)$, the optimal trajectory leads to either of the steady states. While the path to the origin is unique, a continuum of paths leads to the symmetric inner steady state.[5] All of these paths satisfy the conditions of an OL Nash equilibrium but they result in different value functions for each player. This multiplicity of solutions stems from the presence of multiple steady states in a strategic setting, which causes both players to evaluate the paths towards these steady states differently, leading to distinct objective values.

No single solution path is preferred by both players at once; rather, each player's preference is dictated by their own objective function. This is illustrated in Panel 18b, where the objective values are plotted. The crosses indicate the values when both players follow the unique path towards the origin. Since the elite starts with greater power, they attain a higher objective value compared to civil society. The horizontal position of the crosses reflects the initial value of the elite's efforts, with trajectory 1 in Panel 18a representing the path towards the origin. Along this trajectory, the power of both players diminishes continuously. The blue and green curves in Panel 20b illustrate the objective values along the continuum of paths converging to the symmetric inner steady state, representing the value functions of the elite and civil society across different values of $u(0)$. For civil society, objective values decrease as $u(0)$ increases, suggesting that $u(0) = 0$ would be the most advantageous path. Conversely, for the elite, the curve reaches an internal maximum at $u(0) = 4.77$ which significantly surpasses the value of the unique path to the origin and is its preferred solution. These two preferred solutions

[5]Mathematically, this continuum arises from the existence of a three-dimensional stable manifold at the symmetric inner equilibrium. As a result, the canonical system of the OL Nash equilibrium (see Appendix B) has one degree of freedom, represented by the horizontal axis in Panel 18b, which reflects the initial value $u(0)$ with corresponding $v(0)$. Alternatively, fixing $u(0)$ would still allow the same degree of freedom ($v(0)$ correspondingly).

are depicted in Panel 18a, labeled as trajectory 2 (favored by civil society, the maximum of the green curve) and trajectory 3 (favored by the elite, the maximum of the blue curve). While, on trajectory 2, the elite first decreases its capability before it starts to increase along with an increase for civil society, trajectory 3 shows a strong increase in the elite's power followed by a decrease towards the symmetric steady state.

Note that the three plotted OL Nash equilibrium trajectories, along with the continuum of possible solutions, differ from the indifferent solutions at a Skiba point in the optimal control model. In the latter, the elite, acting as a single decision maker, derives the same objective value from different solution paths. In contrast, the OL Nash solution represents a noncooperative strategic outcome from the perspective of both players. Therefore, each solution of the continuum converging to the same steady state is an OL Nash equilibrium, meaning that once the players have committed to this path, any deviation would be suboptimal for both − reflecting the best response property inherent in Nash equilibria. The existence of multiple Nash equilibria, each yielding different payoffs for the players, is akin to the existence of multiple equilibria in simple matrix games with pure strategies. However, gaining an intuitive grasp of this in the context of a differential game requires more reflection. It stems from the fact that the presence of multiple steady states results in different payoffs for both players, each of whom naturally seeks to approach the steady state most favorable to them. Nevertheless, in a Nash equilibrium, each player's optimal strategy is contingent on the strategy of their opponent, necessitating coordination or negotiations between the two players. Although this theoretical result may initially seem abstract, it reflects a realistic scenario where strategic interaction and interdependence shape the outcome.

Comparing this result for the OL Nash game with a high conflict parameter ($b = 1$) with the corresponding result for the one-sided optimization by the elite (or, symmetrically, civil society) in region D (Figure 21 in Appendix C), we see that the game solution allows for interior solutions while the optimization does not do so and results in unproductive outcomes. We interpret this as one advantage of a "competitive" solution (the noncooperative game) with two active agents over a "monopoly" solution with only one active agent. This can be interpreted as analogue in political science to the advantage of competition over monopoly in economics. However, the interior solution (the liberal democracy) is an unstable outcome because of the trajectory to the origin (the possibility of movement into the poverty line solution) and the divergence of the two players' optimal paths into the interior equilibrium.

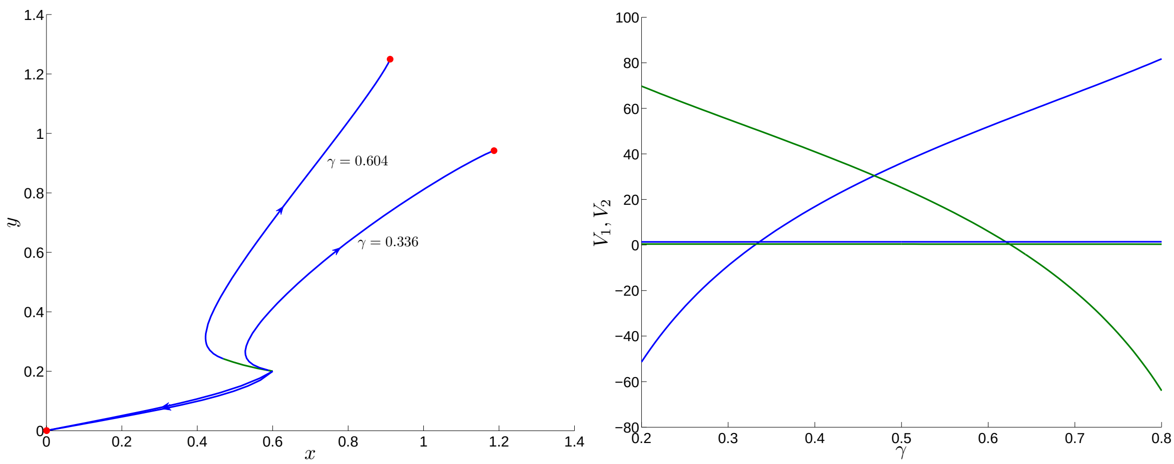
5.2 Cooperative solutions

Cooperative solutions to games assume binding agreements among the players, in our case the elite and civil society. For dynamic games, mostly the weakest solution concept is considered for a cooperative solution, the Pareto solution, also as standard reference point for the noncooperative solution. A sufficient condition for Pareto optimality is the joint maximization of an objective function that is a linear combination of the players' objective functions. In our model, this leads to the optimal control problem

$$\max_{u(t),v(t)} \quad \gamma \int_0^\infty e^{-rt}\left(x^\alpha y^{1-\alpha}\frac{x}{x+y} - \beta u^2\right) \mathrm{d}t + (1-\gamma)\int_0^\infty e^{-rt}\left(x^\alpha y^{1-\alpha}\frac{y}{x+y} - \beta v^2\right) \mathrm{d}t \tag{8}$$

subject to the system equations (5) and (6), where $\gamma \in (0,1)$ is a parameter measuring the weight of the elite versus civil society. Varying γ between 0 and 1 delivers the entire Pareto frontier, that is, the set of (nearly) all Pareto optimal solutions. These solutions are efficient in the sense that no player can be made better off without making the other worse off, but they are no equilibria in the game-theoretic sense. A benevolent government or planner might implement such a solution, which may then be called a "social optimum", notwithstanding general reservations towards this concept.

Figure 19 shows a phase portrait (Panel 19a) and the objective functions (Panel 19b) for initial values of $x_0 = 0.6$ and $y_0 = 0.2$, with weighting parameter γ. For $0.336 < \gamma < 0.604$, the joint maximization of the elite and civil society leads to an interior equilibrium while for smaller or larger γ, the optimal solution leads into the origin. Panel 19a shows the phase portrait for the lower and upper values of γ leading into an interior equilibrium. In Panel 19b, the objective functions of the elite (blue) and civil society (green) are shown for values of γ leading to the interior equilibrium (the increasing objective function value for the elite and the decreasing one for civil society). A Pareto optimum with not too unequal weights given to the two groups (the intermediate range of γ) thus results in a liberal democracy while very high weights for one of the groups leads to an unproductive poverty line society (with the zero objective function value shown by the horizontal line at zero). If we interpret γ as the relative power of the elite and $1-\gamma$ as that of civil society, then an unequal distribution of power can be expected to be detrimental for the society.



(a) Phase portrait.

(b) Objective value functions.

Figure 19: Pareto optimal solution for $x_0 = 0.6$, $y_0 = 0.2$.

If we compare this outcome to the noncooperative solution (Figure 18 above), we see the decisive advantage of cooperation. If both groups act jointly to maximize an objective function that contains each group's preferences with a sufficiently high weight, a unique stable outcome emerges of a liberal democracy. Without cooperation, the liberal democracy (the non-unique interior equilibria) will be fragile and always threatened by the possibility of falling into the equally likely poverty line equilibrium at the origin. This is one way of signifying the inefficiency of the noncooperative equilibrium solution of the dynamic game. The Pareto optimum for intermediate values of the weight parameter γ (the democratic solution) is definitely better for both groups but is no equilibrium solution for the game. Without a binding agreement between both groups, it will not be reached because of the incentive for each of them to attempt unilateral deviation. Strong institutional mechanisms to secure cooperation are required to implement the cooperative outcome, such as a constitutional contract including both groups in the government with sufficient power for each of them. A real-world example is Switzerland's democratic system, where all relevant groups have representatives in the federal government (the Bundesrat) who are obliged to support and defend joint decisions of the body even when they individually would not agree to a particular measure. The remarkable stability of the Swiss democratic system can, at least partly, be explained by this feature of its political institutions.

6 Concluding remarks

Inspired by the work of Acemoglu and Robinson, we analyzed the fragile balance of liberal democracy between anarchy and dictatorship. We used a model of the political process with two groups, the elite and civil society, which uses the Holling type III functional response to describe relations between the groups, and added the possibility of a negative external effect of each group on the other. A Cobb-Douglas production function requiring inputs from both groups determines the output of the society. First, we examined the uncontrolled solution of the model, where both groups react to the other according to the laws of motion only and not by adapting their efforts to influence their own capacities. Using sensitivity and bifurcation analyses, we identified what we called the conflict parameter as having a strong influence on the dynamics of the groups' capacities and on the resulting regime in terms of political institutions. A high conflict parameter implying strong negative effects of each group's actions on the other is a key factor in determining the fragility of a liberal democracy, meaning that inclusive institutions and institutionalized cooperation contribute to a functioning and robust democracy.

Similar results were obtained when we investigated the optimal behavior of the elite as government with civil society adapting to their policies using optimal control theory. Again, high negative effects of the capacities of each group on the other contribute to weakening democracy and putting the system on a path towards anarchy, dictatorship, or a poverty line society. Long-run orientation of the government (a low rate of discounting the future) and not too high an inequality in the initial endowment of the groups also contribute to a stable and robust democracy, although they cannot prevent an unwanted outcome for a society with high potential for conflict. A variety of behavioral patterns was found for the resulting dynamics, including steady states and a limit cycle as a new form of a political business cycle without elections and partisan effects.

Finally, we assumed that both groups follow the aim of maximizing an objective function and act strategically, leading to a dynamic game between them. In the noncooperative case, we characterized a few cases of OL Nash equilibrium solutions, which assume commitment by both groups within an institutional context. At least for sufficiently high and not too unequal initial endowments of both groups, an interior equilibrium is possible, showing that the noncooperative game is more conducive to creating a liberal democracy than the one-sided optimization scenario. The cooperative Pareto optimal solutions of the game, in turn, lead to interior equilibria or an unproductive one, depending on the inequality of the weights given to the two groups. Competitive politics within a framework of appropriate institutions conducive to cooperation seems to provide better chances for a liberal democracy than monopolistic or oligopolistic (noncooperative) politics. Our results impact on political economy by showing that a dynamic model similar to that by Acemoglu and Robinson can explain various dynamics of simple interactions between two groups in society, including switches between liberal democracy, authoritarian dictatorship and anarchy and even cyclical behavior within a democratic society. We identified conditions conducive for a stable liberal democracy in an inclusive state as well as conditions under which such a society may become unstable. The extent of externalities and conflict between the groups and the orientation of policies toward the future are key factors, as is the inequality between the capacities of the two groups.

As Dixit [16] already noted in his review of Acemoglu and Robinson [5], the idea of "narrow corridor" will inspire much further research. From this paper alone, we see research requirements emanating in several directions. Increasing the number of groups is an obvious desideratum, as is the analysis of internal conflicts within any one particular group. More sophisticated game models like FB Nash equilibrium solutions should be explored, despite the difficulties they involve. Alternative models of dynamic systems could also be tried. An empirical test of the results obtained in this paper along lines of the recent work by Acemoglu et al. [1] would be another desideratum. In any case, Acemoglu and Robinson have provided us with many challenges and topics for further research and can be congratulated both for their work and the well-deserved Nobel Memorial Prize in Economic Sciences.

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A Optimality conditions for the optimal control model

In this section we derive the optimality condition of the optimal control model presented in section 4. The government optimizes (4) subject to (5) and (6) over an infinite time horizon.

The current-value Hamiltonian reads

$$\mathcal{H} = x^{\alpha}y^{1-\alpha}\frac{x}{x+y} - \beta u^2 + \lambda^x\left(a\frac{x^2}{c+x^2}\left(1-b\frac{y^2}{c+y^2}\right)u - dx\right) + \lambda^y\left(a\frac{y^2}{c+y^2}\left(1-b\frac{x^2}{c+x^2}\right) - dy\right), \quad (9)$$

where λ^x denotes the adjoint variable (i.e., the dynamic shadow price) of x; λ^y defined accordingly.

Maximization with respect to u implies the first order condition:

$$\mathcal{H}_u = -2\beta u + \lambda^x a\frac{x^2}{c+x^2}\left(1-b\frac{y^2}{c+y^2}\right) = 0$$
$$\implies u = \lambda^x \frac{a}{2\beta}\frac{x^2}{c+x^2}\left(1-b\frac{y^2}{c+y^2}\right). \quad (10)$$

The adjoint variables develop according to the adjoint equations:

$$\dot{\lambda}^x = (r+d)\,\lambda^x - x^{\alpha}y^{1-\alpha}\frac{a\,(x+y)+y}{(x+y)} - \frac{2acx}{(c+x^2)^2}\left(\lambda^x\left(1-b\frac{y^2}{c+y^2}\right)u - \lambda^y b\frac{y^2}{c+y^2}\right), \quad (11a)$$

$$\dot{\lambda}^y = (r+d)\,\lambda^y + x^{\alpha+1}y^{-\alpha}\frac{(\alpha-1)x+\alpha y}{(x+y)^2} + \frac{2acy}{(c+y^2)^2}\left(\lambda^x b\frac{x^2}{c+x^2}u - \lambda^y\left(1-b\frac{x^2}{c+x^2}\right)\right). \quad (11b)$$

The transversality conditions eventually complete the set of optimality conditions:

$$\lim_{t\to\infty} \mathrm{e}^{-rt}\,\lambda^x(t) = 0 \qquad \text{and} \qquad \lim_{t\to\infty} \mathrm{e}^{-rt}\,\lambda^y(t) = 0. \quad (12)$$

B Optimality conditions for the dynamic game

In this section we derive the optimality condition of the open-loop Nash equilibrium solution of the dynamic game presented in section 5. Both players optimize their objective functions (7) subject to the dynamics (1) and (2). Both players assume the strategies of the opponent as exogeneously given implying the Maximum Principle as the appropriate solution concept.

The current value Hamiltonians for both players ($X \dots$ elite, $Y \dots$ civil society) reads

$$\begin{aligned} \mathcal{H}^X &= \frac{x\left(x^\alpha y^{1-\alpha}\right)}{x+y} - \beta u^2 + \xi^x \left(a\frac{x^2}{c+x^2}\left(1-b\frac{y^2}{c+y^2}\right)u - dx\right) \\ &\quad + \xi^y \left(a\frac{y^2}{c+y^2}\left(1-b\frac{x^2}{c+x^2}\right)v - dy\right) \end{aligned} \tag{13a}$$

$$\begin{aligned} \mathcal{H}^Y &= \frac{y\left(x^\alpha y^{1-\alpha}\right)}{x+y} - \beta v^2 + \eta^x \left(a\frac{x^2}{c+x^2}\left(1-b\frac{y^2}{c+y^2}\right)u - dx\right) \\ &\quad + \eta^y \left(a\frac{y^2}{c+y^2}\left(1-b\frac{x^2}{c+x^2}\right)v - dy\right) \end{aligned} \tag{13b}$$

where ξ^x, ξ^y, η^x, and η^y are corresponding adjoint variables.

The first order conditions for the controls of both players we therefore obtain:

$$\begin{aligned} \mathcal{H}^X_u &= -2\beta u + \xi^x a\frac{x^2}{c+x^2}\left(1-b\frac{y^2}{c+y^2}\right) = 0 \\ &\Longrightarrow \quad u = \xi^x \frac{a}{2\beta}\frac{x^2}{c+x^2}\left(1-b\frac{y^2}{c+y^2}\right), \end{aligned} \tag{14a}$$

$$\begin{aligned} \mathcal{H}^Y_v &= -2\beta v + \eta^y a\frac{y^2}{c+y^2}\left(1-b\frac{x^2}{c+x^2}\right) = 0 \\ &\Longrightarrow \quad v = \eta^y \frac{a}{2\beta}\frac{y^2}{c+y^2}\left(1-b\frac{x^2}{c+x^2}\right). \end{aligned} \tag{14b}$$

The adjoint variables for both players develop according to the adjoint equations:

$$\dot{\xi}^x = (r+d)\,\xi^x - x^\alpha y^{1-\alpha}\frac{\alpha\,(x+y)+y}{(x+y)^2} - \frac{2acx}{(c+x^2)^2}\left(\xi^x\left(1-b\frac{y^2}{c+y^2}\right)u - \xi^y\frac{y^2}{c+y^2}bv\right) \tag{15a}$$

$$\dot{\xi}^y = (r+d)\,\xi^y + x^{\alpha+1}y^{-\alpha}\frac{(\alpha-1)\,x+\alpha y}{(x+y)^2} + \frac{2acy}{(c+y^2)^2}\left(\xi^x\frac{x^2}{c+x^2}bu - \xi^y\left(1-b\frac{x^2}{c+x^2}\right)v\right) \tag{15b}$$

$$\dot{\eta}^x = (r+d)\,\eta^x - x^{\alpha-1}y^{2-\alpha}\frac{(\alpha-1)\,x+\alpha y}{(x+y)^2} - \frac{2acx}{(c+x^2)^2}\left(\eta^x\left(1-b\frac{y^2}{c+y^2}\right)u - \eta^y\frac{y^2}{c+y^2}bv\right) \tag{15c}$$

$$\dot{\eta}^y = (r+d)\,\eta^y + x^\alpha y^{1-\alpha}\frac{(\alpha-2)\,x+(\alpha-1)\,y}{(x+y)^2} + \frac{2acy}{(c+y^2)^2}\left(\eta^x\frac{x^2}{c+x^2}bu - \eta^y\left(1-b\frac{x^2}{c+x^2}\right)v\right) \tag{15d}$$

Finally the transversality conditions are given by

$$\begin{aligned} &\lim_{t\to\infty} \mathrm{e}^{-rt}\,\xi^x(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} \mathrm{e}^{-rt}\,\xi^y(t) = 0 \\ &\lim_{t\to\infty} \mathrm{e}^{-rt}\,\eta^x(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} \mathrm{e}^{-rt}\,\eta^y(t) = 0 \end{aligned} \tag{16}$$

C Additional phase portraits

- Starting from Figure 11 (corresponding to region A in the bifurcation diagram 9) we increase the conflict parameter b slightly to $b = 0.056$, we find (for very small $r = 0.005$ and going to region B') the phase portrait of Figure 20. Here there are two horizontally oriented Skiba curves, with trajectories leading to the interior equilibrium as before, either directly or via a detour involving temporarily decreasing, even to zero, the activity of the elite and then increasing it until it eventually decreases again into the interior equilibrium. This looks like a cycle (as in region B) but taking in only one clockwise movement.

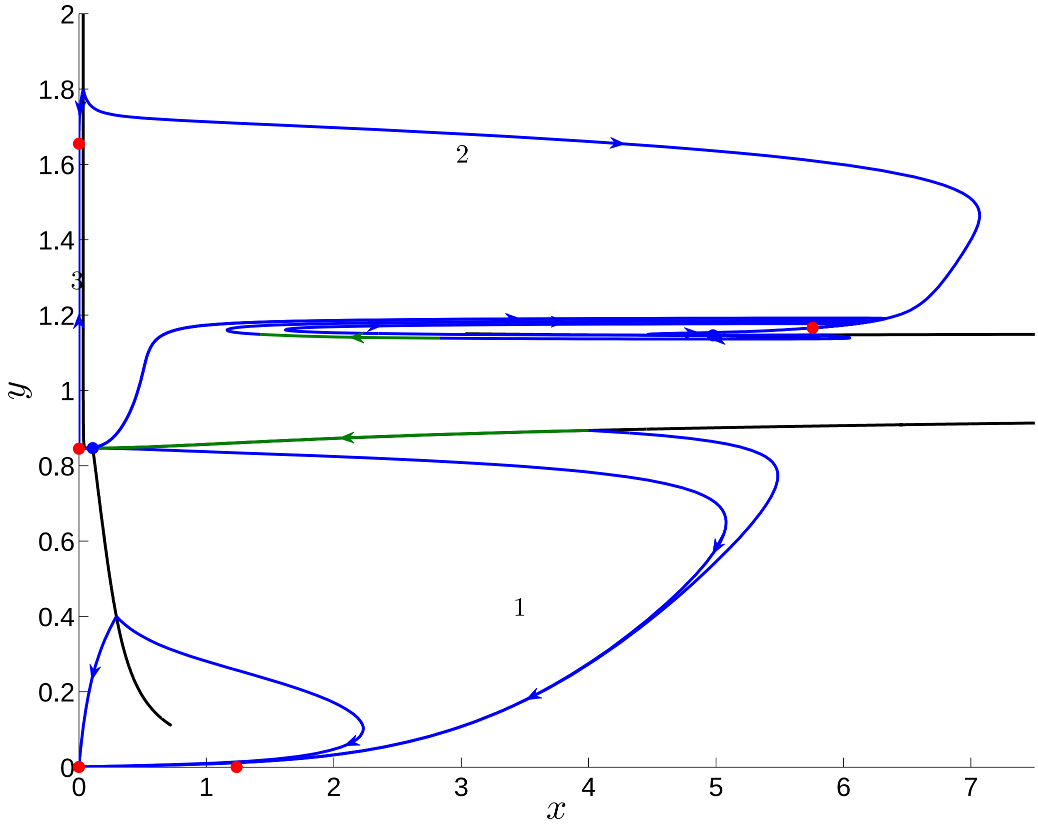

Figure 20: Phase portrait region A, $b = 0.056$, $r = 0.005$.

- Figure 21 continues the discussion of Figure 16 (corresponding to region D in the bifurcation diagram 9) and shows that the situation is essentially the same even for a high conflict parameter and an extremely small rate of discount ($b = 1$, $r = 0.005$). Thus, even a government with long-term planning will result in an unproductive outcome if there is a strong conflict potential and strong externalities between the two groups. If one wants to avoid being trapped in this unproductive situation, institutions avoiding or regulating these political externalities are indispensable. Establishing a system of social partnership as in a corporatist democracy, especially between the private and the public sector, may help to achieve this goal.

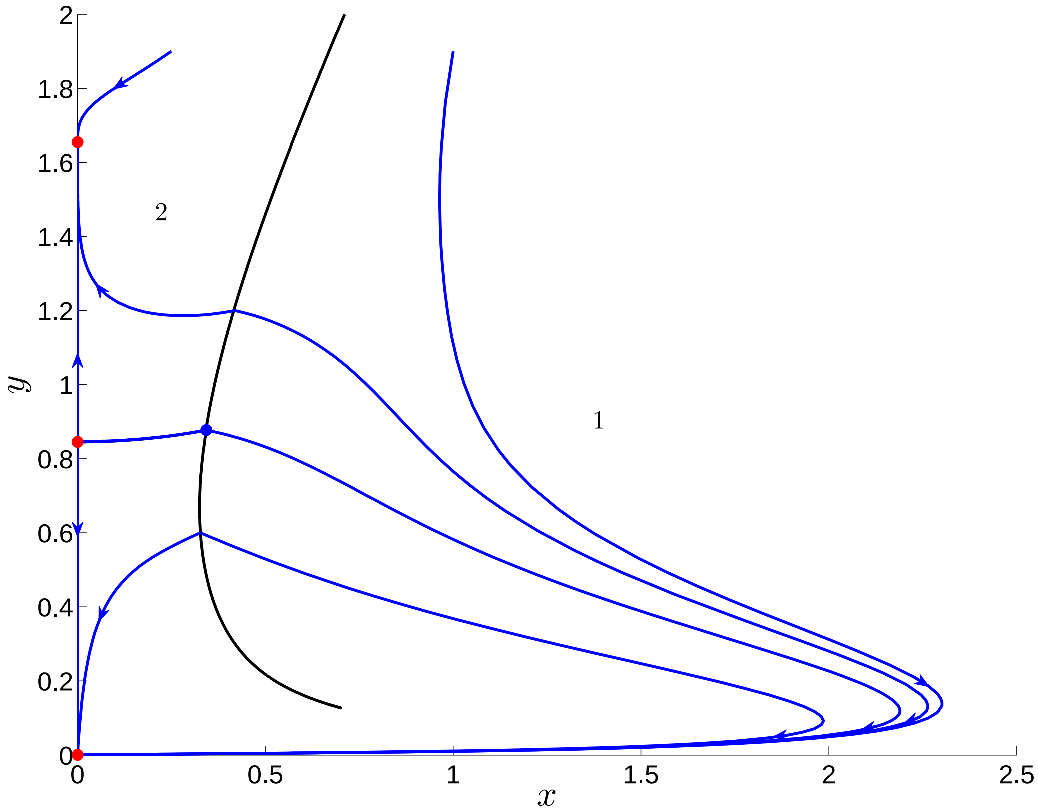

Figure 21: Phase portrait region D, $b = 1$, $r = 0.005$.